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Continuation Theorems and the Approximation-Solvability of Equations Involving Multivalued A-Proper Mappings

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INTRODUCTION

The continuation theorem of Leray and Schauder [21] for a single-valued family of compact displacements is well known to be a powerful tool in establishing existence results for differential and integral equations. Yet its usefulness is limited by its compactness requirement. In recent years this theorem, as well as other results from the classical theory of compact operators, has been extended to various classes of noncompact operators such as monotone, A-proper, and condensing types[1] (see, e.g., [5, 8, 17, 26, 30, 39, 40] for a survey of some of the results in these fields).

It is our purpose in this paper to extend the scope of results of this type to families of multivalued A-proper mappings so as to unify and extend (in a constructive way) the corresponding results for the various mappings mentioned above.

Let X and Y be two real normed spaces with an admissible[1] approximation scheme $\Gamma = \{E_n, V_n; F_n, W_n\}$. In the first part of Section 1 we establish some continuation theorems for families of multivalued mappings T_t $(0 \leqslant t \leqslant 1)$ of $D \subseteq X \to 2^Y$, in which either T_t (for each $t \in [0, 1]$) or T_t (for $t = 1$) is A-proper with respect to Γ, which assert that if T_0 satisfies suitable conditions (which ensure the approximation-solvability of $f \in T_0(x)$), then the equation $f \in T_1(x)$ has the same property. Our conditions on T_0 and our proof of the continuation theorems (Theorems 1.1 to 1.3) and their corollaries employ properties of finite-dimensional Brouwer degree for multivalued maps as described by Ma [22]. Our results extend to multivalued family of A-proper mappings T_t the Leray–Schauder continuation theorem [21] in such a way as to include the recent continuation theorems of Fitzpatrick and Petryshyn [15] for a multivalued ball-condensing family, of Tucker [41] and Milojević [23] for single-valued and

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[1] See Section 1 for precise definitions of the notation, statements of the results, and various contributions mentioned in the Introduction.

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multivalued P_1-compact maps, respectively, of Browder [5] and Nečas [25] (see also [16]) for maps of type (S) and others. We should note that the authors of [15, 41] do not use the degree argument to obtain their continuation theorems but restrict themselves to convex domains. We add that our continuation theorems are related to the homotopy theorems of Browder and Petryshyn [7] for single-valued A-proper maps, of Sadovsky [39] and Nussbaum [26] for single-valued condensing maps, and of Petryshyn and Fitzpatrick [33, 34] for multivalued condensing maps (see also [8, 43]) and of Skrypnik [40] for maps satisfying condition (α).

In the second part of Section 1 we use Theorems 1.1 to 1.3 to establish the approximation-solvability results for equations $f \in T(x)$ and $f \in T(x) + N(x)$, where $T: X \to 2^Y$ and $T + N: X \to 2^Y$ are A-proper and either T is positively homogeneous or $T^{-1}(Q)$ is bounded in X whenever $Q \subset Y$ is relatively compact and where N is required to satisfy suitable growth conditions. In addition to new results, some of the propositions in this subsection include certain recent single-valued results for A-proper mappings due to Petryshyn [29, 31], as well as the subsequent extensions of some of these results to multivalued maps due to Milojević [23].

In the first part of Section 2 we use the results of Section 1 to deduce in a constructuve way certain continuation theorems for single-valued and multivalued families of condensing mappings, P-compact mappings, strongly K-monotone mappings, and mappings of type (KS). The second part of Section 2 is devoted to establishing a number of constructive surjectivity theorems for various special classes of mappings and especially for those of a-stable and of strongly K-monotone type. At each step it is clearly indicated how our results are related to those of other authors, usually obtained by different arguments.

1

Let $\{E_n\}$ and $\{F_n\}$ be two sequences of oriented finite-dimensional spaces and let $\{V_n\}$ and $\{W_n\}$ be two sequences of continuous linear mappings with V_n mapping E_n into X and W_n mapping Y onto F_n, where X and Y are real normed linear spaces.

Remark 1.1. For the sake of notational simplicity we use the same symbol $\| \ \|$ to denote the norms in the respective spaces X, Y, E_n, and F_n and from the context it will be clear which norm is meant. We also use the symbols "$\to$" and "$\rightharpoonup$" to denote *strong* and *weak* convergence, respectively.

DEFINITION 1.1. A quadruple of sequences $\Gamma = \{E_n, V_n; F_n, W_n\}$ is said to be an *admissible scheme* for (X, Y) if $\dim E_n = \dim F_n$ for each n, V_n is injective, $\operatorname{dist}(x, V_nE_n) \to 0$ as $n \to \infty$ for each $x \in X$, and $\{W_n\}$ is uniformly bounded.

Note that in Definition 1.1 we do not require that E_n and F_n be subspaces of X and Y, respectively, nor that V_n and W_n be linear projections. The following examples of admissible schemes, which we subscript for further references, illustrate the generality of Definition 1.1. For the present we assume that $\{X_n\}$ is a sequence of oriented finite-dimensional subspaces of X such that $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each x in X and let V_n be an inclusion map of X_n into X.

(a) Let $\{Y_n\}$ be a sequence of finite-dimensional oriented subspaces of Y such that $\dim Y_n = \dim X_n$ for each n and let Q_n be a continuous linear map of Y onto Y_n such that $\| Q_n \| \leqslant M$ for all n and some $M > 0$. Then $\Gamma_a = \{X_n, V_n; Y_n, Q_n\}$ is admissible for (X, Y).

(b) If $Y = X$, $Y_n = X_n$ and $W_n = P_n$, where P_n is a projection of X onto X_n such that $P_n(x) \to x$ as $n \to \infty$ for each $x \in X$ and $\| P_n \| \leqslant M_0$ for all n, then $\Gamma_b = \{X_n, V_n; X_n, P_n\}$ is an admissible projection scheme for (X, X). Note that when X is complete, then the assumption $\| P_n \| \leqslant M_0$ is superfluous.

(c) If $Y = X^*$, $Y_n = R(P_n^*)$, $V_n = P_n \mid X_n = I_n$ and $W_n = P_n^*$, then $\Gamma_c = \{X_n, P_n; Y_n, P_n^*\}$ is an admissible projection scheme for (X, X^*).

(d) If $Y = X^*$, $Y_n = X_n^*$, and $W_n = V_n^*$, then $\Gamma_d = \{X_n, V_n; X_n^*, V_n^*\}$ is an admissible injection scheme for (X, X^*).

We add that the scheme Γ_d, which proved to be particularly useful (see [5, 20]) for the approximation-solvability of boundary-value problems for differential equations, always exists when X is separable. Example (c) shows that a projection scheme could be admissible for (X, X^*) without being projectionally complete for the pair (X, X^*) (i.e., such that $P_n(x) \to x$ for $x \in X$ and $P_n^*(g) \to g$ for $g \in X^*$).

Let D be a given set in X, $D_n = V_n^{-1}(D)$, $T: D \to 2^Y$ and $T_n = W_n T V_n |_{D_n}$: $D_n \to 2^{F_n}$. The class of multivalued maps T studied in this paper is given by

DEFINITION 1.2. A multivalued map $T: D \subset X \to 2^Y$ is said to be *A-proper* w.r.t. $\Gamma = \{E_n, V_n; F_n, W_n\}$ if $T_n: D_n \to 2^{F_n}$ is upper semicontinuous for each n and if for any sequence $\{u_{n_j} \mid u_{n_j} \in D_{n_j}\}$ such that $\{V_{n_j}(u_{n_j})\}$ is bounded in X and $\| W_{n_j}(y_{n_n}) - W_{n_j}(y)\| \to 0$ as $j \to \infty$ for some $y_{n_j} \in TV_{n_j}(u_{n_n})$ and $y \in Y$, there exists a subsequence $\{u_{n_{j(k)}}\}$ and $x_0 \in D$ such that $V_{n_{j(k)}}(u_{n_{j(k)}}) \to x_0$ and $y \in T(x_0)$.

Remark 1.2. The theory of single-valued A-proper mappings, whose study (via P-compact mappings) was initiated by Petryshyn [28], has also been investigated by a number of other authors, including Browder, Deimling, Fitzpatrick, Grigorieff, Wong and many others (see [30] for other contributors and a survey of the results). The theory proved to be useful in the constructive solvability of differential and integral equations and other fields. It also provided the unification and the extension of various results from the theories of operators of monotone and condensing types. Moreover, there are operators which are A-proper but which are neither of monotone nor of condensing type. Multi-valued A-proper mappings w.r.t. projectionally complete schemes were first extensively studied by Milojević [23] (see also [10] concerning a fixed-point theorem for a multivalued P-compact map).

Given $T: D \to 2^Y$, the *graph* of T, $G(T)$, is defined as $\{(x, y) \mid x \in D, y \in T(x)\}$, the *effective domain* of T is $D(T) = \{x \in D \mid T(x) \neq \varnothing\}$ and the *range* of T, $R(T)$, is defined as $\{y \mid (x, y) \in G(T)\}$.

DEFINITION 1.3. A mapping $T: D \to 2^Y$ is said to be

(1) *locally bounded* at $x_0 \in D$ if there exists a neighborhood N of x_0 such that $T(N \cap D)$ is a bounded subset of Y;

(2) *demiclosed* if $(x_n, y_n) \in G(T)$ for all n, $x_n \to x$ and $y_n \rightharpoonup y$ imply $(x, y) \in G(T)$;

(3) *upper demicontinuous* if for each $x \in D$ and each open half-space V in Y containing $T(x)$, there exists an open neighborhood N of x in X such that $z \in N \cap D$ implies $T(z) \subset V$ (see [12]).

We know that if T is upper semicontinuous (u.s.c.), then T_n is also u.s.c. The following lemma gives some weaker conditions on T that imply upper semicontinuity of T_n (see [23]).

LEMMA 1.1. *Let $T: D \subset X \to 2^Y$. If*

(1) *T is locally bounded and demiclosed on each D_n and Y is reflexive, then T_n is u.s.c. on D_n.*

(2) *T is upper demicontinuous on D and $T(x)$ is nonmepty closed and convex for all x in D, then T is demiclosed.*

The following examples of multivalued mappings, which were proved to be of the A-proper type w.r.t. a given projectionally complete scheme in [23], illustrate the generality of the class of multivalued A-proper mappings. First let us note that if T is A-proper w.r.t. Γ and $C: D \to 2^Y$ u.s.c. and compact (i.e., C maps bounded sets in D into relatively compact sets in Y), then $T + C$ is A-proper w.r.t. T, where we define $(T + C)(x) = \{u + v \mid u \in T(x), v \in C(x)\}$.

In the rest of this paper $C(X)$, $BK(X)$, and $CK(X)$ denote the family of all nonempty compact, bounded closed and convex, and compact and convex subsets of X, respectively.

EXAMPLE 1.1. *If D is a closed subset of X and $C: D \to C(X)$ is u.s.c. and compact, then $T = I - C$ is A-proper w.r.t. Γ_b.*

To be able to state some other examples, we need the following notions.

If A and B are bounded subsets of X, then the Hausdorff distance between A and B is defined as

$$\delta(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

For a bounded subset $A \subset X$ we define the ball-measure of noncompactness (see [39]) of A as $\chi(A) = \inf\{r > 0 \mid A$ can be covered by a finite number of balls of radius less than r with centers in $X\}$.

EXAMPLE 1.2. *If X is complete and $S: X \to CK(X)$ is strictly contractive (i.e., $\delta(S(x), S(y)) \leqslant k \| x - y \|$ for all $x, y \in X$ with $k \in (0, 1)$), then $T = I - S - C$ is A-proper w.r.t. Γ_b provided $\| P_n \| = 1$ for all n.*

EXAMPLE 1.3. *If X is complete and $F: D \subset X \to C(X)$ is u.s.c. and ball-condensing (i.e, for each bounded subset $A \subset D$ with $\chi(A) \neq 0$, $\chi(F(A)) < \chi(A)$), then $T = I - F$ is A-proper w.r.t. Γ_b provided $\| P_n \| = 1$ for all n.*

EXAMPLE 1.4. *Let X be a reflexive Banach space and $T: X \to 2^{X^*}$ demiclosed and strongly monotone, i.e., for each $x, y \in X$, $(u - v, x - y) \geqslant c(\| x - y \|)$ for all $u \in T(x)$, and $v \in T(y)$, where $c: R^+ \to R^+$ is a continuous function such that $c(0) = 0$ and $c(r) > 0$ if $r > 0$. Then T is A-proper with respect to Γ_d and obviously $T(x) \cap T(y) = \varnothing$ if $x \neq y$.*

Proof. The proofs of the claims in Examples 1.1 to 1.3 follow the standard procedure (see [29, 42, 23]). Therefore we restrict ourselves to the proof of the claim of Example 1.4 which has been obtained by Petryshyn [32] for the single-valued case.

Let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ be a bounded sequence such that $V_{n_k}^*(u_{n_k}) - V_{n_k}^*(g) \to 0$ as $k \to \infty$ for some $u_{n_k} \in T(x_{n_k})$ and $g \in X^*$. Then, passing to a subsequence if necessary, we may assume that $x_{n_k} \rightharpoonup x_0$ as $k \to \infty$. From the equality

$$(u_{n_k}, x_{n_k}) = (V_{n_k}^*(u_{n_k}) - V_{n_k}^*(g), x_{n_k}) + (V_{n_k}^*(g), x_{n_k})$$

it follows that $(u_{n_k}, x_{n_k}) \leqslant$ constant. Since $0 \in D(T) = X$, it has been shown by Browder and Hess [6] that $\{u_{n_k}\}$ is bounded. Since $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$, there exists $y_n \in X_n$ such that $y_n \to x_0$ as $n \to \infty$. Hence,

$$(u_{n_k}, x_{n_k} - y_{n_k}) = (V_{n_k}^*(u_{n_k}) - V_{n_k}^*(g), x_{n_k} - y_{n_k}) + (g, x_{n_k} - y_{n_k}) \to 0 \quad \text{as} \quad k \to \infty.$$

But

$$(u_{n_k}, x_{n_k} - x_0) = (u_{n_k}, x_{n_k} - y_{n_k}) + (u_{n_k}, y_{n_k} - x_0) \to 0 \quad \text{as} \quad k \to \infty,$$

since $\{u_{n_k}\}$ is bounded and $y_{n_k} - x_0 \to 0$.

Since $(u_0, x_{n_k} - x_0) \to 0$ for any $u_0 \in T(x_0)$, we see that

$$(u_{n_k} - u_0, x_{n_k} - x_0) \to 0 \qquad \text{as} \qquad k \to \infty.$$

Hence, by the strong monotonicity of T and the properties of the function $c(r)$, we obtain that $x_{n_k} \to x_0$. By the boundedness of $\{u_{n_k}\}$ we may assume that $u_{n_k} \rightharpoonup u_0 \in T(x_0)$, since T is demiclosed. Now let y be any element in X and $y_{n_k} \in X_{n_k}$ such that $y_{n_k} \to y$ as $k \to \infty$. Then $(g - u_0, y) = \lim_k(g - u_{n_k}, y_{n_k}) = \lim_k(V_{n_k}^*(g) - V_{n_k}^*(u_{n_k}), y_{n_k}) = 0$. Since $y \in X$ was arbitrary, $g = u_0 \in T(x_0)$. The fact that $W_n T V_n$ is u.s.c. for each n follows from Lemma 1.1 since T is demiclosed and locally bounded on X (see [38]). This completes the proof of the claim of Example 1.4.

Other examples of (multivalued) A-proper mappings are given below (see also [23]).

Since our proofs of the solvability of operator equations involving multivalued A-proper mappings are based on the degree theory for u.s.c. compact multivalued mappings acting in a finite-dimensional normed space, for the sake of completeness we now state the basic properties of this degree needed in the sequel (see [22]).

Let X_n be a finite-dimensional normed linear space, and let D be a bounded open subset of X_n with a boundary ∂D and closure $\bar{D}$. Let $T: \bar{D} \to CK(X_n)$ be u.s.c. and $p \in X_n$ such that $p \notin T(\partial D)$. Then there exist (see [22]) a single-valued continuous mapping $C: \bar{D} \to X_n$ and an open bounded subset G of X_n such that $p \in G$, $C(x) = x$ for $x \in \bar{D} \setminus G$ and T is homotopic to C; i.e., there exists an u.s.c. multivalued mapping $H: [0, 1] \times \bar{D} \to CK(X_n)$ such that $p \notin H([0, 1] \times \partial D)$, $H(0, x) = T(x)$, and $H(1, x) = C(x)$ for all $x \in \bar{D}$. Then $D \cap G$ is an open bounded subset of X_n and $C: \overline{D \cap G} \to X_n$ is a continuous single-valued mapping with $p \notin \partial(D \cap G)$. Define the degree $\deg(T, D, p)$ to be the Brouwer degree $\deg(C, D \cap G, p)$. As was shown in [22], this definition is independent of the choices of C and G.

For the purposes of this paper we state the following properties of the above degree, whose proofs can be found in [22].

(1) *If* $\deg(T, D, p) \neq 0$, *then* $p \in T(D)$.

(2) *If* $H: [0, 1] \times \bar{D} \to CK(X_n)$ *is u.s.c. in* (t, x) *and* $p \notin H(t, x)$ *for* $t \in [0, 1]$ *and* $x \in \partial D$, *then* $\deg(H_0, D, p) = \deg(H_1, D, p)$.

(3) *If D is convex and symmetric with respect to* 0 *and* $T: \bar{D} \to CK(X_n)$ *is u.s.c. with* $0 \notin T(\partial D)$ *and* $T(x) \cap \lambda T(-x) = \varnothing$ *for all* $x \in \partial D$ *and* $\lambda > 0$, *then* $\deg(T, D, 0)$ *is an odd integer.*

Remark 1.3. It was shown in [34] that (3) holds without convexity of D for odd T. Let us also remark that if T is odd on ∂D with $0 \notin \partial D$, then $T(x) \cap \lambda T(-x) = \varnothing$ for all $x \in \partial D$ and $\lambda > 0$.

To state our results we need the following notion.

DEFINITION 1.4. For a given f in Y, the equation

$$f \in T(x) \qquad (x \in D, f \in Y) \tag{1.1}$$

is said to be *strongly* (resp. *feebly*) *approximation-solvable* w.r.t. Γ if there exists an integer $N_f \geqslant 1$ such that the equation

$$W_n f \in T_n(u) \qquad (u \in D_n\,,\, W_n f \in F_n) \tag{1.2}$$

has a solution $u_n \in D_n$ for each $n \geqslant N_f$ with the property that $V_n(u_n) \to x_0$ in D (resp., $V_{n_j}(u_{n_j}) \to x_0$ for some subsequence $\{u_{n_j}\}$ of $\{u_n\}$) and $f \in T(x_0)$.

Throughout the paper we assume that each A-proper mapping T considered is such that $T_n(u)$ is compact and convex for each $u \in E_n$. This will be so if, e.g., $T(x) \in CK(Y)$ for each $x \in D$ or if $T(x) \in BK(Y)$ for each $x \in D$ with Y reflexive.

Our first result is the following essentially constructive continuation theorem.

THEOREM 1.1. *Let X and Y be normed spaces with an admissible scheme Γ, D a bounded open subset of X with $0 \in D$ and T: $[0, 1] \times \bar{D} \to 2^Y$ such that $T_t = T(t, \cdot)$ is A-proper for each $t \in [0, 1]$, $W_n T V_n$: $[0, 1] \times \bar{D}_n \to CK(F_n)$ is u.s.c. for all n. Suppose that for a given f in Y the following hypotheses hold.*

(H1) *$f \notin T(t, x)$ for $t \in [0, 1]$ and $x \in \partial D$.*

(H2) *$\lambda f \notin T(0, x)$ for $\lambda \in [0, 1]$ and $x \in \partial D$.*

(H3) *If for some $\{t_n\} \subset [0, 1]$ and $x_n \in \partial D_n$ we have that $W_n(f) \in W_n(t_n\,, x_n)$ then there exists $\{t_{n_k}\} \subset \{t_n\}$ such that $t_{n_k} \to t$ and $W_{n_k}(f) - y_{n_k} \to 0$ as $k \to \infty$ for some $y_{n_k} \in W_{n_k} T(t, x_{n_k})$.*

(H4) *There exists an $n_0 \geqslant 1$ such that for some linear isomorphism L_n of F_n onto E_n and $n \geqslant n_0$ we have $\deg(L_n W_n T_0 V_n\,, G_n\,, 0) \neq 0$ with $G_n = V_n^{-1} D_n$.*

Then the equation $f \in T(1, x)$ is feebly approximation-solvable in D w.r.t. Γ. It is strongly-approximation-solvable if $T(1, x) \cap T(1, y) = \varnothing$ whenever $x \neq y$ in $\bar{D}$.

Proof. Let f be a fixed point in Y for which (H1) to (H3) hold. The A-properness of $T(0, \cdot)$ and (H2) imply the existence of an integer n_1 $(\geqslant n_0)$ and a number $\gamma > 0$ (depending on f) such that

$$\| W_n(v) - t W_n(f) \| \geqslant \gamma \qquad \text{for } n \geqslant n_1\,, \quad t \in [0, 1], \quad v \in T(0, V_n(u)), \quad u \in \partial D_n\,, \tag{1.3}$$

where $D_n \equiv V_n^{-1}(D) \equiv \{u \in E_n \mid V_n(u) \in D\}$ and $V_n^{-1}(\bar{D})$ are open and closed sets in E_n, respectively, with $D_n \cap \partial D_n = \varnothing$, $\bar{D}_n \subseteq V_n^{-1}(\bar{D})$, and the boundary $\partial D_n \subset V_n^{-1}(\partial D)$ for each n. Now suppose that our assertion is false. Then there exist a sequence of positive integers $\{n_j\}$ with $n_j \to \infty$ and sequences $\{t_{n_j}\} \subset$

$[0, 1]$ and $u_{n_j} \in \partial D_{n_j}$ with $t_{n_j} \to t_0 \in [0, 1]$ such that for some $v_{n_j} \in T(0, V_{n_j}(u_{n_j}))$, $\| W_{n_j}(v_{n_j}) - t_{n_j} W_{n_j}(f)\| \to 0$ as $j \to \infty$. Hence, by boundedness of $\{W_{n_j}(f)\}$, $W_{n_j}(v_{n_j}) - W_{n_j}(t_0 f) = W_{n_j}(v_{n_j}) - t_{n_j} W_{n_j}(f) + (t_{n_j} - t_0)\, W_{n_j}(f) \to 0$ as $j \to \infty$. Consequently, by the A-properness of $T(0, \cdot)$ w.r.t. Γ, there exist a subsequence $\{u_{n_{j(k)}}\}$ and $x_0 \in \bar{D}$ such that $V_{n_{j(k)}}(u_{n_{j(k)}}) \to x_0$ in X and $t_0 f \in T(0, x_0)$ with $x_0 \in \partial D$, in contradiction to (H2).

Now, since E_n and F_n are finite-dimensional spaces of the same dimension, there is a linear isomorphism L_n of F_n onto E_n. Then, from (1.3) for each $n \geqslant n_0$ we obtain $\| L_n W_n(v) - t L_n W_n(f)\| > 0$ for all $t \in [0, 1]$, $v \in T(0, V_n(u))$ with $u \in \partial D_n$. Consequently, for each $n \geqslant n_1$ the homotopy H_n: $[0, 1] \times \bar{D}_n \to CK(E_n)$ given by $H_n(t, u) = L_n W_n T(0, V_n(u)) - t L_n W_n(f)$ is u.s.c. with $0 \notin H_n(t, u)$ for all $t \in [0, 1]$ and $u \in \partial D_n$. Hence, by homotoly property (3) of the degree for finite-dimensional u.s.c. multivalued mappings, for each $n \geqslant n_1$,

$$\deg(L_n W_n T_0 V_n, D_n, 0) = \deg(L_n W_n T_0 V_n - L_n W_n(f), D_n, 0).$$

Next the A-properness of $T(t, \cdot)$ and (H3) imply the existence of an integer n_2 ($\geqslant n_1$) such that for all $n \geqslant n_2$

$$W_n(f) \notin W_n T(t, V_n(u)) \qquad \text{for all } t \in [0, 1] \quad \text{and} \quad u \in \partial D_n. \tag{1.4}$$

Indeed, if this were not the case, then there would exist sequences $\{n_j\}$ with $n_j \to \infty$ as $j \to \infty$, $\{t_{n_j}\} \subset [0, 1]$ and $\{u_{n_j} \mid u_{n_j} \in \partial D_{n_j}\}$ such that $W_{n_j}(f) \in W_{n_j} T(t_{n_j}, V_{n_j}(u_{n_j}))$ for each n_j. By (H3), there exists $\{t_{n_{j(k)}}\} \subset \{t_{n_j}\}$ such that $t_{n_{j(k)}} \to t$ and $W_{n_{j(k)}}(f) - W_{n_{j(k)}}(v_{n_{j(k)}}) \to 0$ as $k \to \infty$ for some $v_{n_j} \in T(t, V_{n_{j(k)}}(u_{n_{j(k)}}))$. Consequently, by the A-properness of $T(t, \cdot)$ w.r.t. Γ, there exist a subsequence $\{u_m\} \subset \{u_{n_{j(k)}}\}$ and $x_0 \in \bar{D}$ such that $V_m(u_m) \to x_0$ in X as $m \to \infty$ and $f \in T(t, x_0)$ with $x_0 \in \partial D$, in contradiction to (H1). Now, for each $n \geqslant n_2$, define the homotopy G_n: $[0, 1] \times \bar{D}_n \to CK(E_n)$ by $G_n(t, u) = L_n W_n T(t, V_n(u)) - L_n W_n(f)$. Then, as before, using (1.4), we have that $0 \notin G_n(t, u)$ for all $t \in [0, 1]$, $u \in \partial D_n$, and $n \geqslant n_2$. Since G_n is also u.s.c., we obtain for each $n \geqslant n_2$ that

$$\deg(L_n W_n T_1 V_n - L_n W_n(f), D_n, 0) = \deg(L_n W_n T_0 V_n - L_n W_n(f), D_n, 0).$$

Hence, for each $n \geqslant n_2$,

$$\deg(L_n W_n T_1 V_n - L_n W_n(f), D_n, 0) = \deg(L_n W_n T_0 V_n, D_n, 0) \neq 0$$

and consequently, there exists $u_n \in D_n$ such that $L_n W_n(f) \in L_n W_n T(1, V_n(u_n))$ and so, $W_n(f) \in W_n T(1, V_n(u_n))$. By the A-properness of $T(1, \cdot)$ w.r.t. Γ, there exist a subsequence $\{u_{n_j}\}$ and $x_0 \in \bar{D}$ such that $V_{n_j}(u_{n_j}) \to x_0$ in X and $f \in T(1, x_0)$; i.e., the equation $f \in T(1, x)$ is feebly approximation-solvable for this $f \in Y$.

To prove the last assertion of Theorem 1.1, we note that, by what has been proved above, for each $f \in Y$ for which (H1)–(H3) hold there exist a sequence $\{u_n \mid u_n \in D_n\}$ of solutions of $W_n(f) \in W_n T(1, V_n(u_n))$ and a strong limit point x_0 of $\{V_n(u_n)\}$ in $\bar{D}$ such that $f \in T(1, x_0)$. Suppose that $T(1, x) \cap T(1, y) = \varnothing$ whenever $x \neq y$ in $\bar{D}$. Then x_0 is the unique solution of $f \in T(1, x)$ and therefore, $V_n(u_n) \to x_0$ in X. Indeed, if not, then there would exist a subsequence $\{u_{n_k}\}$ such that $\| V_{n_k}(u_{n_k}) - x_0 \| \geqslant \epsilon$ for all k and some $\epsilon > 0$. But $W_{n_k}(f) \in W_{n_k}(1, V_{n_k}(u_{n_k}))$ for each k and therefore, by the A-properness of $T(1, \cdot)$ w.r.t. Γ, there exist a subsequence $\{u_{n_{k(i)}}\}$ and $x_0' \in \bar{D}$ such that $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x_0'$ as $i \to \infty$ and $f \in T(1, x_0')$ with $x_0 \neq x_0'$. This contradiction establishes the last assertion of Theorem 1.1. Q.E.D.

Remark 1.4. If we are interested only in the approximation-solvability of the equation $0 \in T(x)$ $(x \in D)$, then Theorem 1.1 remains valid if we only assume that $T(t, \cdot)\colon \bar{D} \to 2^Y$ is *A-proper at* 0 *for each fixed* $t \in [0, 1]$; i.e., if $\{u_{n_j} \mid u_{n_j} \in \bar{D}_{n_j}\}$ is such that $\{V_{n_j}(u_{n_j})\}$ is bounded in X and $W_{n_j}(u_{n_j}) \to 0$ for some $u_{n_j} \in T(t, V_{n_j}(u_{n_j}))$, then there exist a subsequence $u_{n_{j(k)}}$ and x in $\bar{D}$ such that $V_{n_{j(k)}}(u_{n_{j(k)}}) \to x$ and $0 \in T(x)$.

It is known that even when X is complete and $S\colon \bar{D} \to \bar{D}$ is strictly contractive, then $T = I - S$ is A-proper at 0 but it is unknown whether $T\colon \bar{D} \to X$ is A-proper in the sense of Definition 1.2.

Remark 1.5. Hypothesis (H3) of Theorem 1.1 is implied, for example, when $T(t, x)$ is α-uniformly continuous in t with respect to x in ∂D; i.e., whenever $\{t_n\} \subset [0, 1]$ is such that $t_n \to t$, then

$$\alpha(T(t_n, x), T(t, x)) \equiv \sup_{y \in T(t_n, x)} d(y, T(t, x)) \to 0$$

as $n \to \infty$ uniformly for x in ∂D. Indeed, suppose that for some $t_n \in [0, 1]$ and $u_n \in \partial D_n$ we have that $W_n(f) \in W_n T(t_n, V_n(u_n))$ for all $n \geqslant n_0$. Then $t_{n_k} \to t \in [0, 1]$ for some subsequence $\{t_{n_k}\}$ and $C \equiv \{V_{n_k}(u_{n_k})\} \subset \partial D$. Consequently, $\alpha(T(t_{n_k}, x), T(t, x)) \to 0$ as $k \to \infty$ uniformly for $x \in C$. Since $W_n T(t, V_n(u))$ is compact for each $(t, V_n(u)) \in [0, 1] \times V_n(E_n)$, there exists $y_{n_k} \in T(t, V_{n_k}(u_{n_k}))$ such that $\| W_{n_k}(v_{n_k}) - W_{n_k}(y_{n_k})\| = d(W_{n_k}(v_{n_k}), W_{n_k} T(t, V_{n_k}(u_{n_k})))$ for a fixed $v_{n_k} \in T(t_{n_k}, V_{n_k}(u_{n_k}))$. This implies that $\| W_{n_k}(v_{n_k}) - W_{n_k}(y_{n_k})\| \leqslant \alpha(W_{n_k} T(t_{n_k}, V_{n_k}(u_{n_k})), W_{n_k} T(t, V_{n_k}(u_{n_k}))) \leqslant \| W_{n_k} \| \, \alpha(T(t_{n_k}, V_{n_k}(u_{n_k})), T(t, V_{n_k}(u_{n_k}))) \to 0$ as $k \to \infty$. Since $W_{n_k}(f) = W_{n_k}(v_{n_k})$ for some $v_{n_k} \in T(t_{n_k}, V_{n_k}(u_{n_k}))$, we have $\| W_{n_k}(y_{n_k}) - W_{n_k}(f)\| \to 0$ as $k \to \infty$. This shows that hypothesis (H3) is satisfied.

Analyzing this proof we see that if (H3) is to hold, all we need is that whenever $\{t_n\} \subset [0, 1]$ is such that $t_n \to t$ as $n \to \infty$ then to each $v_n \in T(t_n, x_n)$ with $x_n \in \partial D$ there corresponds $y_n \in T(t, x_n)$ such that $\| v_n - y_n \| \to 0$ as $n \to \infty$. For example, if T, $N\colon \bar{D} \to 2^Y$ are bounded mappings, then the mapping $T_t(x) = T(x) + tN(x)$ for $(t, x) \in [0, 1] \times \bar{D}$ satisfies (H3) by this remark.

Remark 1.6. Analyzing the proof of Theorem 1.1, we see that its assertions are valid if we assume that $T(1, \cdot)$ is A-proper and, instead of (H1)–(H3), we require the following weaker hypotheses.

(H1′) *There exists* $n_f \geqslant 1$ *such that* $W_n(f) \notin W_n T(t, V_n(u))$ *for all* $n \geqslant n_f$, $t \in [0, 1]$ *and* $u \in \partial D_n$.

(H2′) $\lambda W_n(f) \notin W_n T(0, V_n(u))$ *for* $n \geqslant n_f$, $\lambda \in [0, 1]$ *and* $u \in \partial D_n$.

In view of this remark we have the following useful generalization of Theorem 1.1.

THEOREM 1.2. *Let* X, Y *and* D *be as in Theorem* 1.1, $T: [0, 1] \times \bar{D} \to 2^Y$ *such that* $T_1 = T(1, \cdot)$ *is* A*-proper*, $T_n: [0, 1] \times \bar{D}_n \to CK(F_n)$ *u.s.c. for all* n *and for a given* $f \in Y$ *there exists an integer* $n_f \geqslant 1$ *such that hypotheses* (H1′), (H2′), *and* (H4) *hold for all* $n \geqslant n_f$. *Then the conclusions of Theorem* 1.1 *hold.*

An immediate consequence of Theorem 1.2 is the following essentially constructive surjectivity result.

THEOREM 1.3. *Suppose* $T: [0, 1] \times X \to 2^Y$ *is such that* $T(1, \cdot)$ *is* A*-proper w.r.t.* Γ, $W_n T V_n: [0, 1] \times E_n \to C(F_n)$ *is u.s.c. for each* n, *and to any given* f *in* Y *there exist an integer* $n_f \geqslant 1$ *and a number* $r_f > 0$ *such that hypotheses* (H1′), (H2′), *and* (H4) *hold for all* $n \geqslant n_f$ *and* $D = B(0, r_f)$. *Then the equation* $f \in T(1, x)$ *is feebly approximation-solvable w.r.t.* Γ. *It is strongly approximation-solvable if* $T(1, x) \cap T(1, y) = \varnothing$ *whenever* $x \neq y$.

Now we derive a number of special cases of Theorems 1.1, 1.2, and 1.3. We state these results for multivalued mappings and, as seen below, we obtain various new results as well as some known ones on the approximation-solvability of operator equations involving both multivalued and single-valued mappings. We start with two propositions which provide us with conditions on $T(t, x)$ that would imply hypothesis (H4).

PROPOSITION 1.1. *Let* $T: [0, 1] \times X \to 2^Y$ *be such that* $T_1 = T(1, \cdot)$ *is* A*-proper*, $W_n T: [0, 1] \times V_n(E_n) \to CK(F_n)$ *u.s.c. and for each* $f \in Y$, T *satisfies hypotheses* (H1′) *and* (H2′), *and let* $T(0, \cdot)$ *be odd on* $X \backslash B(0, r_f)$. *Then the conclusions of Theorem* 1.3 *hold.*

Proof. All we need show is that hypothesis (H4) holds. Let $f \in Y$ be fixed and let L_n be any linear isomorphism of F_n onto E_n. Since B_n is bounded, convex, and symmetric about $0 \in E_n$ and $L_n W_n T_0 V_n$ is odd on $\partial B_n(0, r_f)$, by finite-dimensional degree property (3) we have that for all $n \geqslant n_f$

$$\deg(L_n W_n T_0 V_n, B_n(0, r_f), 0) \neq 0;$$

i.e., hypothesis (H4) holds. Q.E.D.

PROPOSITION 1.2. *Let* $T: [0, 1] \times X \to 2^Y$ *satisfy all assumptions of Proposition* 1.1 *except for the oddness of* $T(0, \cdot)$. *Let* $K: X \to 2^{Y^*}$ *and* $K_n: E_n \to 2^{F_n^*}$ *be such that* $0 \in K(x)$ *implies* $x = 0$ *and that for each* n *and each* $u \in E_n$ *and* $v \in KV_n(u)$ *there exists* $w \in K_n(u)$ *such that*

(C1) $(g, v) = (W_n g, w)$ *for all* $g \in Y$.

Let M_n *be a linear isomorphism of* E_n *onto* F_n *such that*

(C2) $(M_n u, w) > 0$, *for all* $w \in K_n(u)$ *and* $u \neq 0$ *in* E_n,

and let

(C3) $(v, u) \geqslant 0$ *for all* $v \in T(0, x)$, $u \in K(x)$ *with* $\| x \| \geqslant r \geqslant 0$.

Then the conclusions of Theorem 1.3 *hold.*

Proof. Let $f \in Y$ be fixed. As in the previous proposition, we need only show that (H4) holds. For each $n \geqslant n_f$, consider the homotopy $H_n: [0, 1] \times \bar{B}_n(0, r_f) \to CK(E_n)$ given by $H_n(t, u) = tL_n W_n T(0, V_n(u)) + (1 - t) L_n M_n(u)$, where L_n is as before. We claim that $0 \notin H_n([0, 1] \times \partial B_n(0, r_f))$, $n \geqslant n_f$. If this were not the case, then for some $n \geqslant n_f$ there would exist $u_0 \in \partial B_n(0, r_f)$ and $t_0 \in [0, 1]$ such that $t_0 L_n W_n(v_0) + (1 - t_0) L_n M_n(u_0) = 0$ for some $v_0 \in T(0, V_n(u_0))$. By the injectivity of L_n, we have $t_0 W_n(v_0) + (1 - t_0) M_n(u_0) = 0$. By (H2′), $t_0 \neq 1$, and by the injectivity of M_n, $t_0 \neq 0$. Condition (C1) implies that for each $v \in KV_n(u_0)$ there exists $w \in K_n(u_0)$ such that $(v_0, v) = (W_n(v_0), w) = -((1 - t_0)/t_0)(M_n(u_0), w) < 0$ by condition (C2), in contradiction to (C3). Hence, $0 \notin H_n(t, u)$ for $t \in [0, 1]$ and $u \in \partial B_n(0, r_f)$, $n \geqslant n_f$. By homotopy property (2) we find that for $n \geqslant n_f$

$$\deg(L_n W_n T_0 V_n, B_n(0, r_f), 0) = \deg(L_n M_n, B_n(0, r_f), 0) \neq 0,$$

since $L_n M_n$ is a linear isomorphism of E_n onto itself. Thus, hypothesis (H4) holds. Q.E.D.

We add that the conclusion of Proposition 1.2 holds if instead of (C3) we require

(C3′) $(v, u) \leqslant 0$ for all $v \in T(0, x)$, $u \in K(x)$ with $\| x \| \geqslant r$.

In this case as H_n we take $H_n(t, u) = tL_n W_n T(0, V_n(u)) - (1 - t) L_n M_n(u)$.

In what follows we say that a mapping $T: X \to 2^Y$ satisfies *condition* (+) if $\{x_k\}$ is any sequence and $(x_k, u_k) \in G(T)$ for each k with $u_k \to g$ in Y imply that $\{x_k\}$ is bounded.

Remark 1.7. If $T(0, \cdot)$ is A-proper and satisfies condition (+), then for each $f \in Y$ hypothesis (H2), and hence (H2′), is valid on $\bar{B}(0, r_f)$ for some $r_f > 0$. Indeed, if for some $f \in Y$ hypothesis (H2) fails to hold, then we could find sequences $\{\lambda_n\} \subset [0, 1]$ and $\{x_n\} \subset X$ with $\lambda_{n_j} \to \lambda_0$ and $\| x_{n_j} \| \to \infty$ as $j \to \infty$

such that $\lambda_{n_j} f \in T(0, x_{n_j})$. Since $\lambda_{n_j} f \to \lambda_0 f$ and $\{x_{n_j}\}$ is unbounded, we have that $T(0, \cdot)$ does not satisfy condition $(+)$, a contradiction. Thus, condition $(+)$ implies (H2) and (H2′).

Our first special case of Proposition 1.1 and 1.2 is the following generalization of the First Fredholm theorem.

PROPOSITION 1.3. *Let X and Y be normed spaces and let $A: X \to 2^Y$ be A-proper w.r.t. Γ with $W_n A(x) \in CK(F_n)$ for each $x \in X$. Assume that $A = T + N$, where $T: X \to 2^Y$ is A-proper w.r.t. Γ with $W_n T(x) \in CK(F_n)$ for $x \in X$, positively homogeneous of order $\alpha > 0$ (i.e., $T(tx) = t^\alpha T(x)$ for all $x \in X$ and $t > 0$) and that $0 \in T(x)$ implies $x = 0$, and $N: X \to 2^Y$ is such that*

$$\| y \| / \| x \|^\alpha \to 0 \quad \text{as} \quad \| x \| \to \infty \quad \text{whenever} \quad (x, y) \in G(N). \tag{1.5}$$

Suppose, in addition, that either

(i) *T or A is odd, or*

(ii) *K, K_n, M_n satisfy conditions of Proposition* 1.2 *and either T and K or $T + N$ and K satisfy condition* (C3) *of the same proposition.*

Then the equation

$$f \in T(x) + N(x)$$

is feebly approximation-solvable for each $f \in Y$.

Proof. *Case* (ia). Let $f \in Y$ be fixed. Define a mapping $T_t(x) = T(x) + tN(x)$ for $t \in [0, 1]$ and $x \in X$. Our conditions on T and N imply that there exist an integer $n_f \geqslant 1$ and a number $r_f > 0$ such that for each $n \geqslant n_f$

$$\lambda W_n(f) \notin W_n T_t V_n(u) \qquad \text{for all } \lambda, \quad t \in [0, 1], \quad \text{and} \quad u \in \partial B_n(0, r_f), \tag{1.6}$$

where $B_n = V_n^{-1}(B(0, r_f))$. If not, then there would exist a sequence of positive integers $\{n_j\}$ with $n_j \to \infty$ and sequences $\{u_{n_j} \mid u_{n_j} \in E_{n_j}\}$, $\{\lambda_j\}$, $\{t_j\} \subset [0, 1]$ such that $\| V_{n_j}(u_{n_j})\| \to \infty$ as $j \to \infty$ and $\lambda_j W_{n_j}(f) \in W_{n_j} T_{t_j} V_{n_j}(u_{n_j})$ for all j; i.e., for some $y_j \in TV_{n_j}(u_{n_j})$ and $y_j' \in NV_{n_j}(u_{n_j})$ we have

$$W_{n_j}(y_j) + t_j W_{n_j}(y_j') = \lambda_j W_{n_j}(f), \qquad j \geqslant 1.$$

Since $\{W_n\}$ are linear and uniformly bounded, T is positively homogeneous of order $\alpha > 0$, and $\| V_{n_j}(u_{n_j})\| \to \infty$ as $j \to \infty$, it follows from condition (1.5) and the last equality that

$$\frac{W_{n_j}(y_j)}{\| V_{n_j}(u_{n_j})\|^\alpha} = \frac{\lambda_j W_{n_j}(f) - t_j W_{n_j}(y'_{n_j})}{\| V_{n_j}(u_{n_j})\|^\alpha} \to 0 \qquad \text{as} \qquad j \to \infty.$$

Set $z_{n_j} = u_{n_j}/\| V_{n_j}(u_{n_j})\|$. Since $\{V_{n_j}(z_{n_j})\} \subset X$ is bounded and T is positively homogeneous, we have that $W_{n_j}(\bar{z}_{n_j}) \to 0$ as $j \to \infty$ for

$$\bar{z}_{n_j} = y_{n_j}/\| V_{n_j}(u_{n_j})\|^\alpha \in TV_{n_j}(z_{n_j}).$$

By the A-properness of T, there exist $\{z_{n_{j(k)}}\}$ and $x \in X$ such that $V_{n_{j(k)}}(z_{n_{j(k)}}) \to x$ in X and $0 \in T(x)$ with $\| x \| = 1$, in contradiction to our assumption on T. Hence, relation (1.6) holds for all $n \geqslant n_f$ and consequently, hypotheses (H1′), (H2′), and (H4) hold since, by the oddness of T, we have $\deg(L_n W_n T V_n, B_n(0, r_f), 0) \neq 0$. Thus, all the hypotheses of Theorem 1.3 or Proposition 1.1 are satisfied and consequently the equation $f \in T(x) + N(x)$ is feebly approximation-solvable for each $f \in Y$.

Case (ib). Assume that A is odd and consider the homotopy $T_t'(x) = T(x) + (1 - t) N(x)$. Then, as before, we show that (1.6) holds for the homotopy T_t'. This and the oddness of A imply that all the hypotheses of Proposition 1.1 hold for $T_t'(x)$ and so the conclusion also remains true in this case.

Case (ii). (*a*) Assume that K, K_n, M_n, and T satisfy conditions (C1)–(C3) of Proposition 1.2. Then for $T_t(x)$ defined as in case (ia), we see that all the hypotheses of Proposition 1.2 hold and consequently the assertion of Proposition 1.3 is valid.

(b) Assume that K, K_n, M_n, and $T + N$ satisfy (C1)–(C3). Then for $T_t'(x)$, as defined in case (ib), all the hypotheses of Proposition 1.2 hold, and hence the assertion of Proposition 1.3. Q.E.D.

Remark 1.8. For A single-valued, part (i) of Proposition 1.3 was first proved by Petryshyn [27] and later by Milojevic [23] in the multivalued case.

The following proposition provides us with another set of conditions on T and N which imply the applicability of Theorem 1.3.

PROPOSITION 1.4. *Let X and Y be normed spaces $T\colon X \to 2^Y$ and $N\colon X \to 2^Y$, both bounded such that $T - pN$ is A-proper w.r.t. Γ for each $p \geqslant 1$, $W_n T(x) \in CK(F_n)$, and $W_n N(x) \in CK(F_n)$ for each $x \in X$. Assume that N is odd, $W_n N(x) \subset N(x)$ for each $x \in V_n(E_n)$ and n, $0 \in N(x)$ implies $x = 0$, and that for each $r > 0$ there exists $c_r > 0$ such that $\| y \| \geqslant c_r$ for all $y \in N(x)$ with $\| x \| = r$. Moreover, assume that $T - N$ satisfies condition $(+)$ and that*

($\#$) *there exists $R > 0$ such that $T(x) \cap \lambda N(x) = \varnothing$ for $\| x \| \geqslant R$ and $\lambda \geqslant 1$.*

Then the equation

$$f \in T(x) - N(x)$$

is feebly approximation-solvable for each $f \in Y$.

Proof. Let $f \in Y$ be fixed and define $T_t(x) = N(x) - T(x) + (1-t)f$ for $t \in [0, 1]$ and $x \in X$. Since $T - N$ satisfies condition (+), there exist an $r_f \geqslant R$ and a number $\gamma > 0$ such that

$$\| u - v - \alpha f \| \geqslant \gamma \qquad \text{for all } \alpha \in [-1, 1], \quad u \in N(x), \quad v \in T(x)$$

with

$$\| x \| = r_f . \tag{1.7}$$

By the A-properness of $N - T$ and (1.7), there exist $n_f \geqslant 1$ and a number $\gamma_1 > 0$ such that for each $n \geqslant n_f$

$$\begin{gathered} \| W_n(u) - W_n(v) - sW_n(f) \| \geqslant \gamma_1 \qquad \text{for all } s \in [-1, 1], \quad u \in NV_n(u), \\ \text{and} \quad v \in TV_n(u) \quad \text{with} \quad u \in \partial B_n(0, r_f) = \partial V_n^{-1}(B(0, r_f))). \end{gathered} \tag{1.8}$$

From (1.8) it follows that hypotheses (H1′) and (H2′) of Theorem 1.3 hold. Our assumptions on T and N imply that there exists $n_0 \geqslant n_f$ such that for each $n \geqslant n_0$

$$W_n TV_n(u) \cap \lambda W_n NV_n(u) = \varnothing \qquad \text{for all } u \in \partial B_n(0, r_f) \quad \text{and} \quad \lambda > 1. \tag{1.9}$$

If not, then there would exist infinitely many $\{\lambda_k\} \subset (1, \infty)$ and $u_{n_k} \in \partial B_{n_k}(0, r_f)$ such that

$$W_{n_k}(y_{n_k}) = \lambda_k W_{n_k}(\bar{y}_{n_k}) \qquad \text{for some } y_{n_k} \in TV_{n_k}(u_{n_k}) \quad \text{and} \quad \bar{y}_{n_k} \in NV_{n_k}(u_{n_k}).$$

Since T and N are bounded and $W_n N(x) \subset N(x)$ for $x \in V_n(E_n)$, we have that $M \geqslant \| W_{n_k}(y_{n_k}) \| \geqslant \lambda_k c_{r_f}$. Thus $\lambda_k \leqslant M/c_{r_f}$ for all k and consequently, we may assume that $\lambda_k \to \lambda_0 \in [1, M/c_{r_f}]$. This implies that

$$W_{n_k}(y_{n_k}) - \lambda_0 W_{n_k}(\bar{y}_{n_k}) = (\lambda_k - \lambda_0)\, W_{n_k}(\bar{y}_{n_k}) \to 0 \qquad \text{as} \qquad k \to \infty,$$

since $W_{n_k}(\bar{y}_{n_k}) \in W_{n_k} NV_{n_k}(u_{n_k})$ and N is bounded. By the A-properness of $T - pN$ for $p \geqslant 1$, there exist a subsequence $\{u_{n_{k(i)}}\}$ and $x \in X$ such that $V_{n_{k(i)}}(u_{n_{k(i)}}) \to x$ as $i \to \infty$ and $0 \in T(x) - \lambda_0 N(x)$ with $\| x \| = r_f \geqslant R$, in contradiction to condition (#). Hence, there exists an $n_0 \geqslant n_f$ such that for each $n \geqslant n_0$ relation (1.9) holds. Next, for each $n \geqslant n_0$, consider the homotopy

$$H_t^n(u) = W_n NV_n(u) - tW_n TV_n(u) \qquad \text{for } (t, u) \in [0, 1] \times \bar{B}_n(0, r_f).$$

From (1.8), (1.9), and the properties of N, it follows that $0 \notin H_t^n(u)$ for all $t \in [0, 1]$ and $u \in \partial B_n(0, r_f)$. Since $L_n H_t^n(u)$ is u.s.c. from $[0, 1] \times \bar{B}_n(0, r_f) \to CK(E_n)$ and $0 \notin L_n H_t^n(u)$ for all $(t, u) \in [0, 1] \times \partial B_n(0, r_f)$, where L_n is a linear isomorphism from F_n onto E_n, the homotopy theorem for multivalued u.s.c. maps implies that

$$\deg(L_n W_n NV_n - L_n W_n TV_n, B_n(0, r_f), 0) = \deg(L_n W_n NV_n, B_n(0, r_f), 0)$$

for all $n \geqslant n_f$. By the injectivity of L_n and (1.8), we have that

$$sL_nW_n(f) \notin L_nW_nNV_n(u) - L_nW_nTV_n(u)$$

for all $s \in [-1, 1]$ and $u \in \partial B_n(0, r_f)$. Again by the homotopy theorem,

$$\begin{aligned}
&\deg(L_nW_nT_0V_n, B_n(0, r_f), 0) \\
&\quad = \deg(L_nW_nNV_n - L_nW_nTV_n + L_nW_n(f), B_n(0, r_f), 0) \\
&\quad = \deg(L_nW_nNV_n - L_nW_nTV_n, B_n(0, r_f), 0) \\
&\quad = \deg(L_nW_nNV_n, B_n(0, r_f), 0) \neq 0
\end{aligned}$$

by the oddness of N. Hence, all the hypotheses of Theorem 1.3 hold for $T_t(x)$ and consequently the equation $f \in T(x) - N(x)$ is feebly approximation-solvable. Since f was arbitrary, we have that the equation $f \in T(x) - N(x)$ is feebly approximation-solvable for each $f \in Y$. Q.E.D.

Remark 1.9. We add in passing that condition $(+)$ is implied by any one of the following conditions which have been used by a number of authors (e.g., [5, 13, 20, 27, 36, 38]) in their study of Eq. (1.1) involving single-valued or multivalued mappings of monotone type, ball-condensing type, and A-proper type.

Condition $(1+)$. $(u, v)/\| v \| \to \infty$ as $\| x \| \to \infty$ for all $u \in T(x)$, $v \in Kx$ (i.e., T is *K-coercive*).

Condition $(2+)$. For each unbounded sequence $\{x_n\} \subset X$, $\| u_n \| \to \infty$ as $n \to \infty$ for all $u_n \in T(x_n)$ (i.e., T is *norm-coercive*).

Condition $(3+)$. For each $(x_n, u_n) \in G(T)$ and $(x_n, v_n) \in G(K)$, $\| u_n \| + ((u_n, v_n)/\| v_n \|) \to \infty$ as $\| x_n \| \to \infty$.

Condition $(4+)$. $0 \notin T(\partial B(0, r))$ and $T(tx) = t^\alpha T(x)$ for all $\| x \| \geqslant r$, $t > 1$, and $\alpha > 0$.

When all solutions of $f \in T(x) + tC(x)$ are bounded uniformly for all $t \in [0, 1]$, then we can prove

PROPOSITION 1.5. *Let $T: X \to 2^Y$ satisfy condition $(+)$ and let $C: X \to CK(Y)$ be bounded and such that*

(C4) *$T_t = T + tC$ is A-proper w.r.t. Γ for $t \in [0, 1]$ with $W_nT_t(x) \in CK(F_n)$ for each $x \in X$;*

(C5) *To each $f \in Y$ there corresponds a number $c_f > 0$ such that if the equation $f \in T(x) + tC(x)$ holds for some $x \in X$ and $t \in [0, 1]$, then $\| x \| \leqslant c_f$.*

Assume, in addition, that either one of the following conditions holds.

(i) *There exists* $r > 0$ *such that* T *is odd on* $X \backslash B(0, r)$.

(ii) *Mappings* T, K, K_n, *and* M_n *satisfy conditions* (C1), (C2), *and* (C3) *of Proposition* 1.2.

Then the equation

$$f \in T(x) + C(x)$$

is feebly approximation-solvable for each $f \in Y$.

Proof. Let $f \in Y$ be arbitrary but fixed. Condition (C5) implies that there exists $r_1 > c_f$ such that

$$f \notin T(x) + tC(x) \qquad \text{for all } \| x \| \geqslant r_1 \quad \text{and} \quad t \in [0, 1].$$

The A-properness of T_t for each fixed $t \in [0, 1]$ implies that there exists an $n_0 \geqslant 1$ such that for all $n \geqslant n_0$

$$W_n(f) \notin W_n T V_n(u) + t W_n C V_n(u) \qquad \text{for all } t \in [0, 1], \quad u \in \partial B_n(0, r_1).$$

Since T satisfies condition $(+)$, there exists an $r_2 \geqslant \max\{r, r_1\}$ such that

$$\lambda f \notin T(x) \qquad \text{for all } \lambda \in [0, 1] \quad \text{and} \quad \| x \| = r_2 .$$

Again, by the A-properness of T, there exists $n_1 \geqslant n_0$ such that for all $n \geqslant n_1$

$$\lambda W_n(f) \notin W_n T V_n(u) \qquad \text{for all } \lambda \in [0, 1] \quad \text{and} \quad y \in \partial B_n(0, r_2).$$

Hence, hypotheses (H1′) and (H2′) of Theorem 1.3 hold for the mapping $T_t = T + tC$. Moreover, if condition (i) holds, then the conclusion of our proposition follows from Proposition 1.1, and if condition (ii) holds, the conclusion follows from Proposition 1.2. Q.E.D.

Remark 1.10. Analyzing the proof of Proposition 1.5, we see that it holds if conditions $(+)$ and (C5) are replaced by the following weaker one. *To each* $f \in Y$ *there corresponds* $r_f > 0$ *such that* $\lambda f \notin T_t(x)$ *for all* $t, \lambda \in [0, 1]$ *and* $\| x \| = r_f$.

Remark 1.11. Let us note that condition (C5) of Proposition 1.5 is implied by

Condition (C5′). $\| u_n \| - \| v_n \| \to \infty$ for all $(x_n, u_n) \in G(T)$ and $(x_n, v_n) \in G(C)$ as $\| x_n \| \to \infty$.

The last condition holds, in particular (see also [27]), when for all $(x_n, u_n) \in G(T)$ and all n, $\| u_n \| \to \infty$ as $\| x_n \| \to \infty$ and $| C(x) | \leqslant c(\| x \|) \cdot \| u_n \| + \alpha_2$ with $c: R^+ \to (0, 1 - \epsilon]$ for some $\epsilon > 0$, a continuous function, and $\alpha_2 > 0$, where $| C(x) | = \sup_{v \in C(x)} \| x \|$.

If $\|v_n\|/\|x_n\| \to 0$ as $\|x_n\| \to \infty$ for all $(x_n, v_n) \in G(C)$ and if $\|u\| > \beta\|x\|$ for all $(x, u) \in G(T)$ and all $x \in X$ with $\beta > 0$, then condition (C5′) holds.

Now, in view of the above results, for equations involving a single mapping one easily deduces

PROPOSITION 1.6. *Let X and Y be normed spaces $T: X \to 2^Y$ A-proper w.r.t. Γ and satisfy condition $(+)$. Suppose that there exists $r_0 > 0$ such that either one of the following conditions holds.*

(i) *T is odd on $X \backslash B(0, r_0)$;*

(ii) *Mappings T, K, K_n and M_n satisfy conditions* (C1), (C2) *and* (C3) *of Proposition* 1.2.

Then the equation $f \in T(x)$ is feebly approximation-solvable for each $f \in Y$.

Remark 1.12. For single-valued T and C, Proposition 1.5, part (i), was proved by Petryshyn [31] using the generalized degree theory for A-proper maps. Proposition 1.6 was proved first by Petryshyn [27] when all mappings involved are single-valued, and by Milojević [23] in the present form.

Let us observe that Proposition 1.6 allows us also to study the approximation-solvability of perturbed equations of the form

$$f \in A(x) \equiv T(x) + C(x) \qquad (x \in X, f \in Y), \tag{1.10}$$

where $C: X \to CK(Y)$ is an upper demicontinuous compact operator and T is A-proper w.r.t. Γ. In this case conditions $(+)$ and (C3) of (ii) in Proposition 1.6 are implied, for example, by

ASSUMPTION (a). *T is K*-coercive and $(u, v) \geqslant -c_0\|v\|$ for all $u \in C(x)$, $v \in K(x)$, $x \in X$, and some $c_0 > 0$.

Consequently, under Assumption (a), Eq. (1.10) is feebly approximation-solvable w.r.t. Γ for each $f \in Y$.

We need the following result which was established in Milojević [23] for a projectionally complete scheme and is an extension of the corresponding result of Petryshyn [29] for single-valued mappings. It is easy to check that the proof of this result is valid for any admissible scheme Γ, and hence it is omitted.

PROPOSITION 1.7. *Let X, Y, K, K_n, and M_n be as in Proposition* 1.6. *Let $T: X \to 2^Y$ be A-proper w.r.t. Γ and suppose that for each $f \in Y$ there exists an $r_f > 0$ such that*

(C6) *$(u - f, v) \geqslant 0$ for each $u \in T(x)$, $v \in K(x)$ with $\|x\| = r_f$.*

Then the equation $f \in T(x)$ is feebly approximation-solvable for each $f \in Y$.

Remark 1.13. It is easy to see that if T is K-coercive; i.e., there exists a function $c\colon R^+ \to R^+$ with $c(r) \to \infty$ as $r \to \infty$ and $(u, v) \geqslant c(\| x \|) \| v \|$ for all x in X and all $u \in T(x)$ and $v \in K(x)$, then to each $f \in Y$ there corresponds $n_f > 0$ for which (C6) holds.

2

In the first part of this section we use the results of Section 1 to deduce in a constructive way certain continuation theorems for single-valued and multi-valued families of condensing mappings, P-compact mappings, strongly K-monotone mappings, and mappings of type (KS). The second part of this section is devoted to establishing a number of constructive surjectivity theorems for various special classes of mappings and especially for those of a-stable and of strongly K-monotone type. At each step it is clearly indicated how our results are related to those of other authors, usually obtained by different arguments.

Our first special case of Theorem 1.1 is

COROLLARY 2.1. *Let X be a Banach space with a projectionally complete scheme Γ_b and $\| P_n \| = 1$ for all n, D as in Theorem* 1.1 *and* $F\colon [0, 1] \times \bar{D} \to X$ *continuous and satisfy the following conditions.*

(a) $\chi(F([0, 1] \times Q)) < \chi(Q)$ *whenever* $Q \subset \bar{D}$ *and* $\chi(Q) \neq 0$.

(b) $F(t, x) \neq x$ *for* $x \in \partial D$ *and* $t \in [0, 1]$.

(c) $F(0, x) \neq \lambda x$ *for* $x \in \partial D$ *and* $\lambda > 1$.

Then the equation $F(1, x) = x$ is feebly approximation-solvable w.r.t. Γ_b.

Proof. For each $t \in [0, 1]$, define $T_t = I - F(t, \cdot)$. By Example 1.3, T_t is A-proper for each t and by condition (b), T_t satisfies hypotheses (H1) and (H2) of Theorem 1.1 with $f = 0$. Next we show that T_t satisfies (H3) and (H4) of Theorem 1.1 with $f = 0$. Suppose that for some $t_n \in [0, 1]$ and $x_n \in \partial D_n$ we have that $x_n = P_n F(t_n, x_n)$. Then

$$\chi(\{x_n\}) = \chi(\{P_n F(t_n, x_n)\}) \leqslant \chi(\{F(t_n, x_n)\}) < \chi(\{x_n\}),$$

a contradiction unless $\chi(\{x_n\}) = 0$. Thus, there exist $\{t_{n_k}\} \subset \{t_n\}$ and $\{x_{n_k}\} \subset \{x_n\}$ such that $t_{n_k} \to t$ and $x_{n_k} \to x$ and consequently, $x_{n_k} - P_{n_k} F(t, x_{n_k}) = P_{n_k}(F(t_{n_k}, x_{n_k}) - F(t, x_{n_k})) \to 0$ as $k \to \infty$, by continuity of $F(t, x)$. This proves the validity of (H3). Now, define $H\colon [0, 1] \times \bar{D} \to X$ by $H(t, x) = x - tF(0, x)$. It is clear that H_t is A-proper for each t, continuous in t, uniformly for x in bounded subsets of $\bar{D}$, and $0 \notin H_t(\partial D)$ for $t \in [0, 1]$ by condition (c). Thus, by the homotopy theorem for A-proper mappings [7], the generalized degree

$$\operatorname{Deg}(H_1, D, 0) = \operatorname{Deg}(H_0, D, 0) = \{1\},$$

which implies that there exists an N such that the Brouwer degree $\deg(I - P_n F_0, D_n, 0) = 1$ for all $n \geqslant N$. We have shown that T_t satisfies all the hypotheses of Theorem 1.1 with $f = 0$ and consequently the equation $F(1, x) = x$ is feebly approximation-solvable w.r.t. Γ_b. Q.E.D.

Remark 2.1. Condition (a) of Corollary 2.1 holds, for example, if $F([0, 1] \times \bar{D})$ is relatively compact and so this corollary extends the Leray–Schauder theorem [21]. Corollary 2.1 is also related to the corresponding homotopy results of Sadovsky [39] and Nussbaum [26].

Next we establish Corollary 2.1 for the case of multivalued $F(t, x)$. The reason for treating the multivalued case separately is that, under the upper semicontinuity of $F(t, x)$, hypothesis (H3) does not seem to hold and consequently we are not able to apply Theorem 1.1 in this case. Nevertheless, we are still able to obtain the feeble approximation-solvability of $x \in F(1, x)$ using a variant of Theorem 1.2. To that end we need the following result (see [42] for the single-valued case) which is of interest in its own right.

PROPOSITION 2.1. *Let X be a Banach space with a projectionally complete scheme Γ_b, $\| P_n \| = 1$ for all n, D a bounded open subset of X, and $T: \bar{D} \to CK(X)$ u.s.c. ball-condensing. Suppose that $x \notin T(x)$ for all $x \in \partial D$. Then there exists an integer $N \geqslant 1$ such that* $\deg(I - T, D, 0) = \deg(I - P_n T, D \cap X_n, 0)$ *for all $n \geqslant N$.*

Proof. By the results of [33, 34], $\deg(I - T, D, 0)$ is well defined. We claim that there exists $N_1 \geqslant 1$ such that for all $n \geqslant N_1$ and all $x \in \partial D$ and $t \in [0, 1]$,

$$0 \notin x - tP_n T(x) - (1 - t)\, T(x).$$

If the claim were false, there would exist sequences $\{x_{n_j}\} \subset \partial D$ and $\{t_{n_j}\} \subset [0, 1]$ with $t_{n_j} \to t_0$ and $x_{n_j} - t_{n_j} P_{n_j}(u_{n_j}) - (1 - t_{n_j})\, u_{n_j} = 0$ for some $u_{n_j} \in T(x_{n_j})$. By the boundedness of the set $\{T(x_{n_j}) \mid j \geqslant 1\}$, we have

$$x_{n_j} - t_0 P_{n_j}(u_{n_j}) - (1 - t_0)\, u_{n_j} = (t_{n_j} - t_0)\,(P_{n_j}(u_{n_j}) - u_{n_j}) \to 0.$$

Then, since $\chi(\{P_{n_j}(u_{n_j})\}) \leqslant \chi(\{u_{n_j}\})$,

$$\begin{aligned}\chi(\{t_0 P_{n_j}(u_{n_j}) + (1 - t_0)\, u_{n_j}\}) &\leqslant t_0 \chi(\{P_{n_j}(u_{n_j})\}) + (1 - t_0)\, \chi(\{u_{n_j}\}) \\ &\leqslant \chi(\{u_{n_j}\}) \leqslant \chi(\{Tx_{n_j}\}) < \chi(\{x_{n_j}\}).\end{aligned}$$

Hence,

$$\chi(\{x_{n_j}\}) \leqslant \chi(\{t_0 P_{n_j}(u_{n_j}) + (1 - t_0)\, u_{n_j}\}) < \chi(\{x_{n_j}\}),$$

a contradiction unless $\chi(\{x_{n_j}\}) = 0$. By passing to a subsequence, we may assume that $x_{n_j} \to x_0 \in \partial D$. This and the u.s.c. of T imply that $0 \in x_0 - T(x_0)$, a contradiction. Thus our claim is valid. Next, since $I - T$ is A-proper w.r.t.

Γ_b and $x \notin T(x)$ for $x \in \partial D$, it is easy to see that there exists an $N_2 \geqslant 1$ such that $x \notin P_n T(x)$ for all $x \in \partial D \cap X_n$ and all $n \geqslant N_2$. Set $N = \max\{N_1, N_2\}$. Then $\deg(I - P_n T, D \cap X_n, 0)$ is defined for all $n \geqslant N$.

Now for each $n \geqslant N$, define the mapping

$$H(t, x) = tP_n T(x) + (1 - t)\, T(x), \qquad x \in D, \quad t \in [0, 1].$$

It is clear that for each $t \in [0, 1]$, $H(t, \cdot)$ is ball-condensing and $x \notin H(t, x)$ for all $x \in \partial D$ and $t \in [0, 1]$ as shown above. Moreover, for $Q \subset D$ with $\chi(Q) \neq 0$. $\chi(H([0, 1] \times Q)) \leqslant (1 - t)\, \chi(T(Q)) < \chi(Q)$, which, by Lemma 3.3 and Theorem 2.2 of Petryshyn and Fitzpatrick [34], implies that

$$\deg(I - H_0, D, 0) = \deg(I - H_1, D, 0)$$

or $\deg(I - T, D, 0) = \deg(I - P_n T, D, 0)$ for all $n \geqslant N$. By the excision theorem for compact multivalued mappings [22], we have that

$$\deg(I - P_n T, D, 0) = \deg(I - P_n T, D \cap X_n, 0). \qquad \text{Q.E.D.}$$

COROLLARY 2.1′. *Let X, Γ_b, and D be as in Corollary 2.1, F: $[0, 1] \times \bar{D} \to CK(X)$ u.s.c. and satisfy condition (a) of Corollary 2.1, and*

(b′) *$x \notin F(t, x)$ for $x \in \partial D$ and $t \in [0, 1]$,*

(c′) *$\lambda x \notin F(0, x)$ for $x \in \partial D$ and $\lambda > 1$.*

Then the equation $x \in F(1, x)$ is feebly approximation-solvable w.r.t. Γ_b.

Proof. As before, define $T_t = I - F(t, \cdot)$ for $t \in [0, 1]$. By Lemma 3.3 and Theorem 2.2 of Petryshyn and Fitzpatrick [34], $\deg(T_0, D, 0) = \deg(T_1, D, 0)$, which, in view of Proposition 2.1, implies the existence of an integer $N \geqslant 1$ such that

$$\deg(I - P_n F_1, D \cap X_n, 0) = \deg(I - P_n F_0, D \cap X_n, 0)$$

for all $n \geqslant N$.

Now, let H: $[0, 1] \times \bar{D} \to CK(X)$ be defined by $H(t, x) = tF(0, x)$. By condition (c′), $x \notin H(t, x)$ for all $x \in \partial D$ and $t \in [0, 1]$ and, by the above-mentioned results of Petryshyn and Fitzpatrick [34], $\deg(I - F_0, D, 0) = \deg(I, D, 0) = 1$. Thus, for each $n \geqslant N$,

$$\begin{aligned} \deg(I - P_n F_1, D \cap X_n, 0) &= \deg(I - P_n F_0, D \cap X_n, 0) \\ &= \deg(I - F_0, D, 0) \neq 0. \end{aligned}$$

This implies that there exists $x_n \in D \cap X_n$ such that $x_n \in P_n F(1, x_n)$ for all $n \geqslant N$. By the A-properness of F_1, there exists $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x$ and $x \in F(1, x)$. Q.E.D.

In a more general setting, Corollary 2.1′ was proved by Fitzpatrick and Petryshyn [15] for convex D using only an ingenious retraction-type argument. Using the fixed-point index of Fitzpatrick and Petryshyn [15] for multivalued condensing mappings, Milojević [23] obtained the existence part of Corollary 2.1′.

The following corollary was obtained by Milojević [23], using the fixed-point index of Fitzpatrick and Petryshyn [15] for multivalued condensing mappings, and by Tucker [41] in the single-valued case, using a retraction argument, for the case when $D = B(0, r)$ (see also [11]).

COROLLARY 2.2. *Let X be a normed space with a projectionally complete scheme Γ_b, $D \subset X$ open and bounded with $0 \in D$ and $F: [0, 1] \times \bar{D} \to 2^X$ such that*

(a) *F_t is P_1-compact at 0 for each $t \in [0, 1]$ and $P_n T: [0, 1] \times \bar{D}_n \to CK(X_n)$ u.s.c. for all n.*

(b) *$x \notin F(t, x)$ for $x \in \partial D$ and $t \in [0, 1]$.*

(c) *$\lambda x \notin F(0, x)$ for $x \in \partial D$, $\lambda \geqslant 1$ and F_0 bounded.*

(d) *If for some $x_n \in \partial D_n$ and $t_n \in [0, 1]$ we have $x_n \in P_n T(t_n, x_n)$, then there exists $\{t_{n_k}\} \subset \{t_n\}$ such that $t_{n_k} \to t$ and $x_{n_k} - P_{n_k}(y_{n_k}) \to 0$ for some $y_{n_k} \in F(t, x_{n_k})$.*

Then the equation $x \in F(1, x)$ is feebly approximation-solvable w.r.t. Γ_b.

Proof. Let $T_t = I - F_t$ for $t \in [0, 1]$. It follows from our conditions (b) to (d) that T_t satisfies hypotheses (H1)–(H3) of Theorem 1.1 with $f = 0$. Next we prove that T_0 satisfies hypothesis (H4) of Theorem 1.1 with $f = 0$. To that end, we claim that there exists $N \geqslant 1$ such that $\lambda x \notin P_n F(0, x)$ for all $x \in \partial D_n$, $\lambda \geqslant 1$, and $n \geqslant N$. If not, then there would exist $x_{n_k} \in \partial D_{n_k}$ and $\lambda_{n_k} \geqslant 1$ such that $\lambda_{n_k} x_{n_k} \in P_{n_k} F(0, x_{n_k})$ for each $k \geqslant 1$. Since F_0 is bounded and $\| x_{n_k} \| \geqslant \delta$ for some $\delta > 0$, we see that $\{\lambda_{n_k}\}$ is bounded. Thus, we may assume that $\lambda_{n_k} \to \lambda \geqslant 1$. Then $\lambda x_{n_k} - \lambda_{n_k} x_{n_k} \to 0$ as $k \to \infty$ and so, by P_1-compactness at 0 of F_0, it follows that there exists $x \in \partial D$ such that $\lambda x \in F(0, x)$, in contradiction to (c). Thus, an integer N with the above property does exist.

Now for each $n \geqslant N$, define $H_n: [0, 1] \times \bar{D}_n \to CK(X_n)$ by $H_n(t, x) = x - t P_n F(0, x)$. Then $0 \notin H_n(t, x)$ for all $x \in \partial D_n$ and $t \in [0, 1]$ and, by finite-dimensional homotopy property (2), we see that

$$\deg(I - P_n F_0, D_n, 0) = \deg(I, D_n, 0) = 1$$

for all $n \geqslant N$. This proves that T_0 satisfies (H4) with $f = 0$. Now it remains to apply Theorem 1.1 and Remark 1.4. Q.E.D.

To continue with our special cases, we need

DEFINITION 2.1. Let X and Y be real Banach spaces, $D \subset X$ a closed subset of X, and K a mapping from X into 2^{X^*}. Then a mapping T from D into 2^Y is said to be

(1) *K-monotone* if for all $x, y \in D$ there exists $f \in K(x - y)$ such that $(u - v, f) \geqslant 0$ whenever (x, u) and (y, v) belong to $G(T)$;

(2) *strongly K-monotone* if $(u - v, f) \geqslant c(\| x - y \|) \| g \|$ for some f, $g \in K(x - y)$ and all (x, u) and (y, v) in $G(T)$, where $c: R^+ \to R^+$ is a continuous function such that $c(0) = 0$ and $c(r) > 0$ if $r > 0$;

(3) of *type* (KS) if

(a) the set $T(x) \subset Y$, is nonempty, bounded, closed and convex for each x in D;

(b) for each finite-dimensional subspace F of X, T is upper semi-continuous from $D \cap F$ into the weak topology of Y (i.e., to a given element $x_0 \in D \cap F$ and a weak neighborhood V of $T(x_0)$ in Y, there exists a neighborhood U of x_0 in F such that $T(x) \subset V$ for all $x \in D \cap U$);

(c) for each $\{x_n\} \subset D$ with $x_n \rightharpoonup x$ in X and $(u_n, f_n) \to 0$ for some $u_n \in T(x_n)$ and $f_n \in K(x_n - x)$ we have that $x_n \to x$ in X;

(4) of *type* (KS_+) if conditions (a) and (b) of part (3) hold and

(c') for each $\{x_n\} \subset D$ with $x_n \rightharpoonup x$ in X and $\limsup (u_n, f_n) \leqslant 0$ for some $u_n \in T(x_n)$ and $f_n \in K(x_n - x)$ we have that $x_n \to x$ in X.

Monotone mappings ($K = I$, $Y = X^*$) were introduced independently by Vainberg–Kachurovsky and Zarantonello, and further studied by Minty, Browder, Kachurovsky, Rockefellar, and others (see [5, 17] for references). The study of J-monotone mappings (J a duality mapping) was initiated by Browder, while those of the single-valued K-monotone type with K a suitable mapping, by Kato and Petryshyn. Single-valued mappings of types (S) and (S_+) from X into X^* ($K = I$) were introduced and studied by Browder [2], and those of modified type (KS) and (KS_+) by Petryshyn [35].

DEFINITION 2.2. We say that a mapping $T: D \to 2^Y$ is *K-quasi-bounded* if for any bounded sequence $\{x_n\} \subset D$, $y_n \in T(x_n)$ and $f_n \in K(x_n)$ with $(y_n, f_n) \leqslant c \| x_n \|$ for each n and some constant $c > 0$, the sequence $\{y_n\}$ is bounded in Y.

In addition to bounded mappings, it was shown by Browder and Hess [6] that monotone mappings ($K = I$, $Y = X^*$) with 0 in the interior of its effective domain are quasi-bounded. We add that using similar arguments, one can show that a K-monotone mapping defined on X is K-quasi-bounded provided that K is linear.

Let us continue the discussion of our results in this section by introducing some new classes of A-proper mappings.

PROPOSITION 2.2. *Let X, Y be reflexive Banach spaces, $D \subset X$ closed, $\Gamma_a = \{X_n, V_n; Y_n, Q_n\}$ an admissible scheme, $T: D \to 2^Y$. Let $K: Y \to Y^*$ be a bounded mapping such that*

(a_1) *$Kx = 0$ implies $x = 0$, K is positively homogeneous of order $\alpha > 0$, and the range $R(K)$ is dense in Y^*;*

(a_2) *for each $x \in X_n$ and $g \in Y$, we have that $(Q_n(g), Kx) = (g, Kx)$;*

(a_3) *K is weakly continuous at 0 and is uniformly continuous on closed balls in X.*

Then, if $T: D \to 2^Y$ is K-quasi-bounded, demiclosed, and of type (KS), T is A-proper w.r.t. Γ_a.

Proof. Let $\{x_{n_j} \mid x_{n_j} \in D_{n_j}\}$ be a bounded sequence such that for some $g \in Y$, $Q_{n_j}(u_{n_j}) - Q_{n_j}(g) \to 0$ as $j \to \infty$ in Y for some $u_{n_j} \in T(x_{n_j})$. Then, in view of (a_2) and the equality

$$(u_{n_j}, K(x_{n_j})) = (Q_{n_j}(u_{n_j}) - Q_{n_j}(g), K(x_{n_j})) + (Q_{n_j}(g), K(x_{n_j})),$$

the sequence $\{(u_{n_j}, K(x_{n_j}))\}$ is bounded, and consequently, $\{u_{n_j}\}$ is bounded by the K-quasi-boundedness of T. By the reflexivity of X, we may assume that $x_{n_j} \rightharpoonup x_0$ and since $\operatorname{dist}(x_0, X_n) \to 0$, there exists $y_n \in X_n$ such that $y_n \to x_0$ in X. Let $B(0, r)$ be a ball in X that contains x_0, $\{x_{n_j}\}$, and $\{y_{n_j}\}$. Since $x_{n_j} - y_{n_j} \rightharpoonup 0$, by ($a_2$) and ($a_3$) and the weak continuity of K at 0,

$$\begin{aligned}&(u_{n_j}, K(x_{n_j} - y_{n_j}))\\ &\quad = (Q_{n_j}(u_{n_j}) - Q_{n_j}(g), K(x_{n_j} - y_{n_j})) + (Q_{n_j}(g), K_{n_j}(x_{n_j} - y_{n_j})) \to 0\end{aligned}$$

as $j \to \infty$.

Now, since $K(tx) = t^\alpha K(x)$ for $x \in X$ and $t > 0$, and $\{x_{n_j}\}$, $\{y_{n_j}\}$, and x_0 lie in $\bar{B}(0, r)$ for each j, we have the relation

$$\begin{aligned}&(u_{n_j}, K(x_{n_j} - x_0))\\ &\quad = (u_{n_j}, K(x_{n_j} - y_{n_j})) + 2^\alpha(u_{n_j}, K(\tfrac{1}{2}(x_{n_j} - x_0)) - K(\tfrac{1}{2}(x_{n_j} - y_{n_j})))\end{aligned} \tag{2.1}$$

with $\frac{1}{2}(x_{n_j} - x_0)$ and $\frac{1}{2}(x_{n_j} - y_{n_j})$ lying in $\bar{B}(0, r)$. For each $t > 0$, define the function $\psi(t)$ as in [18] by

$$\psi(t) = \sup\{\| Kx - Ky \| \mid \| x - y \| \leqslant t, x, y \in \bar{B}(0, r)).$$

Since K is uniformly continuous on $\bar{B}(0, r)$, the function $\psi(t)$ is nondecreasing in t, $\psi(t) \to 0$ as $t \to 0$ and

$$\| Kx - Ky \| \leqslant \psi(\| x - y \|) \qquad \text{for } x, y \quad \text{in} \quad \bar{B}(0, r). \tag{2.2}$$

Since $\frac{1}{2}(x_{n_j} - x_0)$ and $\frac{1}{2}(x_{n_j} - y_{n_j})$ lie in $\bar{B}(0, r)$, $\{u_{n_j}\}$ is bounded by some constant c and $\| y_{n_j} - x_0 \| \to 0$, it follows from (2.2) that $(u_{n_j}, K(\frac{1}{2}(x_{n_j} - x_0)) - K(\frac{1}{2}(x_{n_j} - y_{n_j}))) \mid \leqslant c\psi(\frac{1}{2} \| y_{n_j} - x_0 \|) \to 0$ as $j \to \infty$. In view of (2.1), this and the fact that $(u_{n_j}, K(x_{n_j} - y_{n_j})) \to 0$ imply that $(u_{n_j}, K(x_{n_j} - x_0)) \to 0$ as $j \to \infty$. Since T is of type (KS), $x_{n_j} \to x_0 \in D$. By the boundedness of $\{u_{n_j}\}$, we may assume that $u_{n_j} \rightharpoonup u_0 \in T(x_0)$ since T is demiclosed.

Now, let $y \in X$ be arbitrary and $y_n \in X_n$ such that $y_n \to y$. Then, by assumption (a_3) on K,

$$(u_0 - g, K(y)) = \lim(u_{n_j} - g, K(y_{n_j})) = \lim(Q_{n_j}(u_{n_j}) - Q_{n_j}(g), K(y_{n_j}))$$
$$= (0, K(y)).$$

Hence, since $R(K)$ is dense in Y^* and $(u_0 - g, w) = 0$ for each $w \in R(K)$, it follows that $g = u_0 \in T(x_0)$. Finally, the upper semicontinuity of $Q_n T \mid D_n$ follows from Definition 2.1(b). Q.E.D.

Remark 2.2. Analyzing the proof of Proposition 2.2, we see that $T: D \to 2^Y$ is A-proper at 0 w.r.t. Γ_a without requiring the weak continuity of K at 0.

Since (a) of Definition 2.1 was not used in the proof of Proposition 2.2, as an immediate corollary of it, Lemma 1.1 and Proposition 1.7, we have the following result provided that $\| Kx \| \geqslant c_0 > 0 \ \forall \| x \| = 1$.

COROLLARY 2.2. *Let X, Y, Γ_a, and K be as in Proposition 2.2 and $D \subset X$ closed and convex. Then a K-quasi-bounded locally bounded, and demiclosed mapping $T: D \to 2^Y$, which is strongly K-monotone, is A-proper w.r.t. Γ_a. In particular, if $D = X$ and $c(r) \to \infty$ as $r \to \infty$, then $f \in T(x)$ is strongly approximation-solvable for each f in Y.*

Remark 2.3. The local boundedness of T in Corollary 2.2 was used just to show that $Q_n T \mid D_n$ is u.s.c. Thus, it can be replaced by any other assumption that implies the u.s.c. of $Q_n T \mid D_n$, $n \geqslant 1$.

We say that Banach space X has *property* (H) if it is strictly convex and if $x_n \rightharpoonup x$ in X and $\| x_n \| \to \| x \|$ imply that $x_n \to x$ in X.

COROLLARY 2.3. *Let X, Y, Γ_a, and K be as in Proposition 2.2 with X having property* (H). *Let $T: X \to 2^Y$ be K-quasi-bounded locally bounded, demiclosed, and such that for each x, y in X $(u - v, K(x - y)) \geqslant (\psi(\| x \|) - \psi(\| y \|)) (\| x \| - \| y \|)$ for $u \in T(x)$, $v \in T(y)$, where $\psi: R^+ \to R^+$ is continuous, strictly increasing, and $\psi(r) \to \infty$ as $r \to \infty$. Then T is A-proper w.r.t. Γ_a.*

Proof. Under property (H) of X, we see that T satisfies condition (c) of Definition 2.1 and consequently Proposition 2.2 is applicable. Q.E.D.

Let us note that a duality mapping $J: X \to 2^{X^*}$ with respect to a gauge function ψ is an example of mappings treated in Corollary 2.3 ($Y = X^*$). We add that Proposition 2.2 and its corollaries were proved by Petryshyn [29, 35] in the single-valued case.

We are now in a position to state another particular case of our continuation Theorem 1.1 for multivalued mappings of type (KS).

COROLLARY 2.4. *Let X, Y, Γ_a, and K be as in Proposition 2.2, $D \subset X$ open and bounded with $0 \in D$, $T: [0, 1] \times \bar{D} \to 2^Y$ with $T_1 = T(1, \cdot)$ K-quasi-bounded, demiclosed, and of type (KS), and $Q_nT: [0, 1] \times \bar{D}_n \to CK(Y_n)$ u.s.c. for all $n \geqslant 1$. Suppose that for $f \in Y$ there exists an integer $n_f \geqslant 1$ such that for each $n \geqslant n_f$ the following conditions hold.*

(H1′) $Q_n(f) \notin Q_nT(t, x)$ *for all* $t \in [0, 1]$ *and* $x \in \partial D_n$;

(H2′) $\lambda Q_n(f) \notin Q_nT(0, x)$ *for all* $\lambda \in [0, 1]$ *and* $x \in \partial D_n$;

(H4) $\deg(L_nQ_nT_0, D_n, 0) \neq 0$ *for some linear isomorphism* L_n *of* Y_n *onto* X_n.

Then the equation $f \in T(1, x)$ is feebly approximation-solvable w.r.t. Γ_a.

Remark 2.4. As we have already noted in Section 1, hypotheses (H1′) and (H2′) hold if

(a) T_t is K-quasi-bounded, demiclosed, and of type (KS) for each $t \in [0, 1]$ and α-uniformly continuous in t for x in bounded subsets of D;

(b) $f \notin T(t, x)$ for all $t \in [0, 1]$ and $x \in \partial D$;

(c) $\lambda f \notin T(0, x)$ for all $t \in [0, 1]$ and $x \in \partial D$.

Remark 2.5. We have seen in Section 1 (Propositions 1.1 and 1.2) that hypothesis (H4) holds if either

(i) T_0 is odd on ∂D and D is symmetric about 0, or

(ii) there exist mappings $K: X \to Y^*$ and $K_n: X_n \to Y_n^*$ such that $K(x) = 0$ implies $x = 0$ and for each $n \geqslant 1$,

(C1) $(g, K(x)) = (Q_n(g), K_n(x))$ for all $x \in D_n$ and $g \in Y$,

(C2) $(M_n(x), K_n(x)) > 0$ for all $x \in \partial D_n$ and some linear isomorphism M_n of X_n onto Y_n,

(C3) either $(u, K(x)) \geqslant 0$ or $(u, K(x)) \leqslant 0$ for all $u \in T_0(x)$ with $x \in \partial D$.

In view of Remark 2.2, if we are interested in the approximation-solvability of $0 \in T_1(x)$ in Corollary 2.4, then we can dispense with the assumption of the weak continuity of K at 0. This fact and Remarks 2.4 and 2.5(i) imply the validity of the following result analogous to Corollary 2.4.

COROLLARY 2.5. *Let X, Y, Γ_a, K, and D be as in Corollary 2.2, except that K*

is not assumed to be weakly continuous at 0; $T: [0, 1] \times \bar{D} \to Y$ *such that the following conditions hold.*

(a) T_t *is K-quasi-bounded, demiclosed, and of type* (KS) *for each* $t \in [0, 1]$, *and* T_t *continuous in t uniformly for x in bounded subsets of* $\bar{D}$.

(b) $T_t(x) \neq 0$ *for all* $t \in [0, 1]$ *and* $x \in \partial D$.

(c) T_0 *is odd on* ∂D.

Then the equation $T_1(x) = 0$ *is feebly approximation-solvable w.r.t.* Γ_a.

Corollary 2.5 improves an existence result of Nečaš [25], who treats the case when $Y = X^*$, $K = I$, and T_t is a bounded demicontinuous mapping of type (S_+) for $t \in [0, 1]$ (however, no separability of X was required in [25]). We also add that a result of Nečaš [25] is an extension of an earlier result of Browder [5], who assumed the separability of X, and the boundedness and the continuity of T_t for $t \in [0, 1]$. For other extensions and applications see [16].

Remark 2.6. In view of our results in Section 1, we see that conditions on K and T in Corollary 2.2 together with, for example, K-coerciveness of T, imply the strong approximation-solvability of the equation $f \in T(x)$ for each $f \in Y$ and, in particular, the surjectivity of T. However, if somehow we can establish the surjectivity of T without the restrictive assumption (a_3) on K in Proposition 2.2, then we can establish the A-properness of T and thus the approximation-solvability of $f \in T(x)$, as we see from the following generalization of Theorem 2 of Petryshyn [28].

PROPOSITION 2.3. *Let X and Y be normed linear spaces with an admissible scheme* $\Gamma_a = \{X_n, V_n; Y_n, Q_n\}$. *Let* $T: X \to 2^Y$ *be surjective, lower semi-continuous on X, u.s.c. from each* X_n *to the weak topology on Y, and approximation-stable; i.e., for all sufficiently large n we have*

$$\|Q_n(u_n) - Q_n(v_n)\| \geqslant c(\|x - y\|)$$

whenever (2.3)

$$x, y \in X_n, \qquad u_n \in T(x), \qquad v_n \in T(y),$$

where $c: R^+ \to R$ *is a continuous function,* $c(0) = 0$ *and* $c(r) > 0$ *for* $r > 0$. *Then T is A-proper w.r.t.* Γ_a.

Proof. Let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ be a bounded sequence such that $Q_{n_j}(u_{n_j}) - Q_{n_j}(g) \to 0$ as $j \to \infty$ in Y for some $u_{n_j} \in T(x_{n_j})$ and $g \in Y$. By the surjectivity of T, there exists $x_0 \in X$ such that $g \in T(x_0)$. Since $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$, there exists $y_n \in X_n$ such that $y_n \to x_0$. Then, by the lower semicontinuity of T, we can choose $v_n \in T(y_n)$ with $v_n \to g$.

From this and (2.3) we obtain that

$$\begin{aligned} &c(\| x_{n_j} - y_{n_j} \|) \\ &\qquad \leqslant \| Q_{n_j}(u_{n_j}) - Q_{n_j}(v_{n_j})\| \\ &\qquad \leqslant \| Q_{n_j}(u_{n_j}) - Q_{n_j}(g)\| + \| Q_{n_j} \| \cdot \| g - v_{n_j} \| \to 0 \qquad \text{as} \qquad j \to \infty. \end{aligned}$$

This implies that $\| x_{n_j} - y_{n_j} \| \to 0$. Indeed, if $\| x_{n_j} - y_{n_j} \| \to a > 0$, by the continuity of c, $c(\| x_{n_j} - y_{n_j} \|) \to c(a) > 0$, a contradiction. Thus, $x_{n_j} \to x_0$ with $g \in T(x_0)$. That $Q_n T \mid X_n$ is u.s.c. follows from the upper semicontinuity of T from X_n to the weak topology on Y. Q.E.D.

Remark 2.7. Let X, Y, and Γ_a be as in Proposition 2.3 and $T: X \to 2^Y$ surjective, lower semicontinuous on X, u.s.c. from each X_n to the weak topology on Y and strongly K-monotone with $K: X \to 2^{Y^*}$ bounded, $(g, u) = (Q_n(g), u)$ for all $g \in Y$ and $u \in K(x)$ with $x \in X_n$ and $\| u \| = \| x \|$ for each $u \in K(x)$ and $x \in X$.

Then it is easy to see that T satisfies inequality (2.3) and consequently, by Proposition 2.3, T is A-proper w.r.t. Γ_a. Thus Proposition 1.7 implies that if, for example, $c(r) \to \infty$ as $r \to \infty$, then $f \in T(x)$ is strongly approximation-solvable for each $f \in Y$.

Analyzing the proof of Proposition 2.3, we see that for mappings defined on subsets of X, the following result is valid.

PROPOSITION 2.4. *Let X, Y, Γ_a, and c be as in Proposition 2.3, $D \subset X$ closed and $T: D \to 2^Y$ lower semicontinuous and u.s.c. from $D_n = D \cap X_n$ to the weak topology on Y for each $n \geqslant 1$. Moreover, suppose that for a given $g \in Y$, the equation $g \in T(x)$ is solvable in D and T is approximation-stable relative to D; i.e., for all sufficiently large n we have*

$$\| Q_n(u_n) - Q_n(v_n)\| \geqslant c(\| x - y \|)$$

whenever

$$x, y \in D_n, \qquad u_n \in T(x), \qquad v \in T(y).$$

Then T is A-proper at g w.r.t. Γ_a.

As an immediate consequence of Proposition 2.3 and Theorem 1.3 we obtain the following approximation-solvability result.

THEOREM 2.1. *Let X and Y be normed linear spaces with an admissible scheme Γ_a, $T: [0, 1] \times X \to 2^Y$ such that $Q_n T: [0, 1] \times X_n \to CK(Y_n)$ is u.s.c. for every n, $T_1 = T(1, \cdot)$ is lower semicontinuous on X, surjective and T_1 is approximation-stable.*

Suppose, in addition, that for each $f \in Y$ there exist an integer $n_f \geqslant 1$ and a number $r_f > 0$ such that for each $n \geqslant n_f$ hypotheses (H1′), (H2′), *and* (H4) *of Theorem* 1.3 *hold. Then the equation $f \in T(1, x)$ is strongly approximation-solvable for each $f \in Y$.*

Similarly, in view of Proposition 2.4 and Theorem 1.1, a continuation theorem for the strong approximation-solvability of $0 \in T(1, x)$ in $D \subset X$ for this type of mappings can be stated.

Accretive Mappings

Now we apply our results to operator equations involving mappings of (strongly) J-monotone type. First, we introduce some new notations. Recall that the normalized duality mapping $J: X \to 2^{X^*}$ is defined by $J(0) = 0$ and for $x \neq 0$, $J(x) = \{w \in X^* \mid (w, x) = \| w \| \cdot \| x \|, \| w \| = \| x \|\}$. For each pair of $x, y \in X$, define $(x, y)_+ = \sup\{(w, x) \mid w \in J(y)\}$. Then we define a somewhat more general type of mapping than those of J-monotone type, to which we refer as accretive mappings.

DEFINITION 2.3. A mapping $T: D \subset X \to X$ is said to be *accretive* if $(T(x) - T(y), x - y)_+ \geqslant 0$ for every $x, y \in D$, and *strongly accretive* if $(T(x) - T(y), x - y)_+ \geqslant c(\| x - y \|) \| x - y \|$ for every $x, y \in D$, where $c: R^+ \to R^+$ is continuous with $c(0) = 0$ and $c(r) > 0$ for $r > 0$.

Let D be a convex and closed subset of X. We say that $T: D \to X$ satisfies *condition* (I) if $\operatorname{dist}(x - \lambda T(x), D) = o(\lambda)$ as $\lambda \to 0^+$ for every $x \in D$. It follows immediately from the duality formula (see, for example, [9])

$$\operatorname{dist}(z, D) = \max\{x^*(z) - \sup_D x^*(y) \mid x^* \in X^*, \| x^* \| = 1\} \qquad \text{for } z \in X$$

that condition (I) is equivalent to

"If $x \in D$, $\quad x^* \in X^* \backslash \{0\}$ $\quad$ and $\quad x^*(x) = \sup\{x^*(y) \mid y \in D\}$,
then $\quad x^*(-Tx) \leqslant 0$".

For $D = \bar{B}(0, r)$ condition (I) is equivalent to

$$(-Tx, x)_+ \leqslant 0 \qquad \text{for each } x \in \partial B(0, r). \tag{2.4}$$

Indeed, suppose that condition (I) holds and take $x \in \partial B(0, r)$. Then for any $x^* \in J(x)$ we have that $x^*(x) = \sup\{x^*(y) \mid y \in \bar{B}(0, r)\}$ which, by condition (I), implies that $x^*(-Tx) \leqslant 0$. Since x^* was arbitrary, we have that $(-Tx, x^*)_+ \leqslant 0$. Conversely, suppose that condition (2.4) holds. If $x_0 \in B(0, r)$, then condition (I) always holds. Now suppose that $x_0 \in \partial B(0, r)$ and that $x^* \in X^* \backslash \{0\}$ with $x^*(x_0) = \sup\{x^*(y) \mid y \in \bar{B}(0, r)\}$. We need only show that $x^*(-Tx_0) \leqslant 0$. Since $\| x^* \| = (1/r)\, x^*(x_0)$, we have that $x^*(x_0) = \| x_0 \| \cdot \| x^* \|$. Let $k > 0$

such that $k \| x^* \| = r$ and define $x_k^*(x) = kx^*(x)$ for $x \in X$. Then $\| x_k^* \| = r = \| x_0 \|$ and $x_k^*(x_0) = k \| x_0 \| \cdot \| x^* \| = \| x_k^* \| \cdot \| x_0 \|$ and consequently, $x_k^* \in J(x_0)$. By condition (2.4) we have that $x_k^*(-T(x_0)) \leqslant 0$ and so $x^*(-(Tx_0)) \leqslant 0$. Thus condition (I) holds. In view of this remark, we have the following constructive extension of an existence result of Deimling [9] for $D = \bar{B}(0, r)$.

THEOREM 2.2. *Let X be a Banach space with a projectionally complete scheme $\Gamma_b = \{X_n, V_n; X_n, P_n\}$, $\| P_n \| = 1$, and $T: \bar{B}(0, r) \subset X \to X$ continuous and such that $(Tx - Ty, x - y) \geqslant C(\| x - y \|) \| x - y \|$ and $(-Tx, x)_+ \leqslant 0$ for each $\| x \| = r$. Then the equation $Tx = 0$ is uniquely approximation solvable w.r.t. Γ_b.*

Proof. Since $P_n^* J(x) \subset J(x)$ for each $x \in X_n$, we have that our condition on T implies that $(-P_n T(x), x)_+ \leqslant 0$ for every $x \in \partial B_n(0, r)$ and $n \geqslant 1$. Thus T and $P_n T$ satisfy condition (I) on $\bar{B}(0, r)$ and $\bar{B}_n(0, r)$, respectively, and, by [9, Theorem 2], the equations $T(x) = 0$ and $P_n T(x) = 0$ are uniquely solvable in $\bar{B}(0, r)$ and $\bar{B}_n(0, r)$, respectively, for each $n \geqslant 1$. By Proposition 2.4, T is A-proper at 0, which, together with the injectivity of T, implies that $x_n \to x$ and $T(x) = 0$, where $x_n \in \bar{B}_n(0, r)$ are such that $P_n T(x_n) = 0$. Q.E.D.

To obtain the unique approximation solvability of $T(x) = f$ for each $f \in X$, we need the following surjectivity result for accretive mappings.

In what follows we say that $T: X \to Y$ satisfies condition $(++)$ provided that whenever $\{x_n\} \subset X$ is a bounded sequence such that $Tx_n \to g$ for some g in Y, then there is $x \in X$ such that $Tx = g$.

THEOREM 2.3. *Let X be a real Banach space with a projectionally complete scheme $\Gamma_b = \{X_n, V_n; X_n, P_n\}$ and $\| P_n \| = 1$. Suppose that $T: X \to X$ is continuous and accretive. Then, if T satisfies conditions $(+)$ and $(++)$, $T(X) = X$.*

Proof. Let $f \in X$ be fixed. Since the equation $T(x) = f$ is solvable if and only if the equation $T_1(x) = f - T(0)$ is solvable with $T_1(x) = T(x) - T(0)$, and T_1 has the same properties as T, we may assume that $T(0) = 0$.

For each positive integer n, define $T_n(x) = T(x) + (1/n)\, x$, $x \in X$. Then T_n is strongly accretive with $c(r) = (1/n)\, r$, since $(T_n(x) - T_n(y), x - y)_+ = (1/n) \| x - y \|^2 + (T(x) - T(y), x - y)_+ \geqslant (1/n) \| x - y \|^2$ for all $x, y \in X$.

Consequently, by Deimling's theorem [9], T_n is surjective and by Proposition 2.3, T_n is A-proper w.r.t. Γ_b for each n.

Now define $K = J$, $K_n = P_n^* J: X_n \to X_n' = R(P_n^*) \subset X^*$ and $M_n = I_n$ on X_n. Since $P_n^* J(x) \subset J(x)$ for all $x \in X_n$, we see that K, K_n, and M_n satisfy all the hypotheses of [31, Theorem 4] (see [24]). Moreover, for each $x \in X$, $(T(x), x)_+ \geqslant 0$ and $(x, u) = \| x \|^2 \geqslant 0$ for all $u \in J(x)$. Thus, by [31, Theorem 4], which is easily seen to be valid when the condition $(T(x), u) \geqslant 0$, $u \in J(x)$ is replaced by $(T(x), x)_+ \geqslant 0$, T is surjective, i.e., $T(X) = X$. Q.E.D.

For the strongly accretive mappings we have the following constructive result.

THEOREM 2.4. *Let X and Γ_b be as in Theorem 2.3 and $T: X \to X$ continuous and strongly accretive. Then, if T satisfies condition $(+)$, the equation $Tx = f$ is uniquely approximation-solvable for each $f \in X$.*

Proof. Let $f \in X$ be fixed. Since $T(x) = f$ is uniquely approximation-solvable if and only if $T_1(x) = f - T(0)$ is uniquely approximation-solvable with $T_1(x) = T(x) - T(0)$, we may assume, as before, that $T(0) = 0$. By Proposition 2.3 and Theorem 2.3, T is A-proper w.r.t. Γ_b. Then T, $K = J$, $K_n = P_n{}^*J$, and $M_n = I_n$ satisfy all the hypotheses of Proposition 1.6 except $(T(x), u) \geqslant 0$ for $u \in J(x)$, $\| x \| \geqslant r_0$. However, it is easy to check that Proposition 1.6 is valid if $(T(x), u) \geqslant 0$, $u \in J(x)$, is replaced by $(T(x), x)_+ \geqslant 0$, $x \in X$. Thus, the equation $T(x) = f$ is uniquely approximation solvable for each $f \in X$. Q.E.D.

Remark 2.8. We have seen that condition $(+)$ is implied by various other conditions and we mention now only two:

(1) $\| Tx \| \to \infty$ *as* $\| x \| \to \infty$

(2) $\liminf_{r\to\infty} c(r) > 0$, *where* $c(r)$ *is as in Definition* 2.3.

Let us prove that (2) implies condition $(+)$. Suppose that $T(x_n) \to f$ as $n \to \infty$. Then, for each $\epsilon > 0$, there exists $n(\epsilon) \geqslant 1$ such that $c(\| x_n - x_m \|) \leqslant \| T(x_n) - T(x_m)\| < \epsilon$ for all $n, m \geqslant n(\epsilon)$. Let $m_0 \geqslant n(\epsilon)$ be fixed. Then $\| x_n - x_{m_0} \| \leqslant M$ for all $n \geqslant n(\epsilon)$ and some constant M, for otherwise

$$0 < \liminf_{n\to\infty} c(\| x_n - x_{m_0} \|) \leqslant \liminf_{n\to\infty} \| T(x_n) - T(x_{m_0})\| < \epsilon,$$

a contradiction, since ϵ can be chosen arbitrarily small. Hence, $\{x_n - x_{m_0}\}$ is bounded and consequently, $\{x_n\}$ is bounded, proving that T satisfies condition $(+)$.

COROLLARY 2.6. *Let X and Γ_b be as in Theorem 2.3 and $T: X \to X$ continuous and strongly accretive. Then, if either $\| Tx \| \to \infty$ as $\| x \| \to \infty$ or $\liminf_{n\to\infty} c(r) > 0$, the equation $T(x) = f$ is uniquely approximation solvable for each $f \in X$.*

Remark 2.9. Corollary 2.6 for the case when $\liminf c(r) > 0$ as $r \to \infty$ was proved by Deimling [9]. Under the additional assumptions that $c(r)$ is strictly increasing, $c(r) \to \infty$ as $r \to \infty$, X is reflexive, X^* is strictly convex, and J is weakly continuous, it was proved earlier by Petryshyn (see [30]). Under the assumption that X^* is uniformly convex, it was proved by Browder [3] without the requirement that J be weakly continuous.

Remark 2.10. When $T: X \to Y$ is Gateaux differentiable, then one may use in Theorem 2.1 the following surjectivity result of Pohožaev [37].

Let X and Y be Banach spaces with Y reflexive (uniformly convex, respectively) and suppose that $T: X \to Y$ is continuously Gateaux differentiable with $T(X)$ weakly closed in Y (Gateaux differentiable with $T(X)$ closed in X, resp.). Let $dT_x^*: Y^* \to X^*$ denote the adjoint mapping of the Gateaux derivative dT_x of T at a point x. Then, if the null space $N(dT_x^*) = \{0\}$ for each $x \in X$, $T(X) = Y$. Let us also add that certain surjectivity results for strongly K-monotone mappings of Browder [4] and Kirk [19] can also be used in conjunction with Theorem 2.1 to obtain the unique approximation-solvability of equations involving such mappings (see also [35]).

Let us now turn our attention to the approximation-solvability of equations of the form $T(x) - M(x) = f$ with T condensing (for example) and M linear. We start with some preliminaries. Let X and Y be normed linear spaces and let $\{X_n\}$ be a sequence of finite-dimensional subspaces of X such that $\operatorname{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. Let $M: X \to Y$ be a linear bijection such that for each $y \in Y$ and $Z_n = M(X_n)$, $\operatorname{dist}(y, Z_n) \to 0$ as $n \to \infty$. Let $\{Y_n\}$ be another sequence of finite-dimensional subspaces of Y with $\dim Z_n = \dim Y_n$ and let $Q_n: Y \to Y_n$ be a continuous linear projection for each n with $\{Q_n\}$ uniformly bounded. Denote by V_n and M_n injections of X_n into X and of Z_n into Y, respectively. Then we have

PROPOSITION 2.5. *Let X, Y, and M be as above with M also continuous. Let $T: X \to BK(Y)$ be bounded such that $pI - TM^{-1}$ is A-proper w.r.t. $\Gamma = \{Z_n, M_n; Y_n, Q_n\}$ for each $p \geqslant 1$. Assume that $T - M$ satisfies condition $(+)$ and that there exists $R > 0$ such that*

$$T(x) \cap \lambda M(x) = \varnothing \qquad \text{for } \|x\| \geqslant R \text{ and } \lambda \geqslant 1. \tag{2.5}$$

Then the equation

$$f \in M(x) - T(x) \tag{2.6}$$

is feebly approximation-solvable with respect to $\{X_n, V_n; Y_n, Q_n\}$ for each $f \in Y$.

Proof. Let $f \in Y$ be fixed and for each $n \geqslant 1$ consider the approximate equation

$$Q_n(f) \in Q_n M(x_n) - Q_n T(x_n), \qquad x_n \in X_n. \tag{2.7}$$

Set $y_n = M(x_n)$. Then from Eq. (2.7) we obtain

$$Q_n(f) \in Q_n(I - TM^{-1})(y_n). \tag{2.8}$$

It follows that Eq. (2.7) is solvable in X_n if and only if Eq. (2.8) is solvable in Z_n. Consequently, the equation $f \in M(x) - T(x)$ is feebly approximation-solvable with respect to $\{X_n, V_n; Y_n, Q_n\}$ if and only if the equation $f \in y - TM^{-1}y$ is feebly approximation-solvable with respect to $\{Z_n, M_n; Y_n, Q_n\}$. In view of this fact, it remains to show that $f \in y - TM^{-1}y$ is feebly approximation-solvable with respect to $\{Z_n, M_n; Y_n, Q_n\}$. To that end, we show that all hypotheses of Proposition 1.4 are satisfied for TM^{-1} and I. It is clear that our boundary condition (2.5) implies $\lambda y \notin TM^{-1}(y)$ for $\| y \| \geqslant R/K$ and $\lambda \geqslant 1$, where $\| M^{-1}g \| \geqslant k \| y \| y$. It remains to show that $I - TM^{-1}$ satisfies condition $(+)$. Let $\{y_n\}$ be a sequence in Y such that $y_n - u_n \to g$ for some $u_n \in TM^{-1}(y_n)$ and $g \in Y$. Then setting $x_n = M^{-1}(y_n)$ for each n, it follows that $Mx_n - u_n \to g$ for some $u_n \in T(x_n)$. Since $T - M$ satisfies condition $(+)$ on X, the sequence $\{x_n\}$ is bounded and so is $\{u_n\}$; moreover, for some constant $c > 0$, $\| Mx_n - u_n \| \leqslant c$ for all n. Consequently, by the boundedness of T, $\| y_n \| = \| M(x_n) \| \leqslant \| u_n \| + c$ is uniformly bounded since $\{u_n\}$ is bounded. Q.E.D.

Remark 2.11. Condition (2.5) of Proposition 2.5 is implied by the following condition.

(c) *There exists a constant* $c > 0$ *such that if* $0 \in Mx - tTx$ *for some* $x \in X$ *and* $t \in [0, 1]$, *then* $\| x \| \leqslant c$.

Indeed, if condition (2.5) were not satisfied, then there would exist $\{x_n\} \subset X$ with $\| x_n \| \to \infty$ and $\lambda_n \geqslant 1$ such that $\lambda_n M(x_n) \in T(x_n)$ for all n from which, in view of (c), it follows that $\| x_n \| \leqslant c$, a contradiction.

It is clear that, (see also [24]) if we assume (c) in Proposition 2.5, then its assertion remains valid if the continuity of M is replaced by the continuity of M^{-1}: $Y \to X$.

Remark 2.12. Proposition 2.5 certainly holds if we choose $Y_n = Z_n$. Assuming additionally that Y is complete and $T: X \to CK(Y)$ is bounded, u.s.c., and ball-condensing, and $\| M^{-1} \| \leqslant 1$, then for each $p \geqslant 1$ the map $pI - TM^{-1}$ is A-proper w.r.t. $\Gamma = \{Z_n, M_n; Z_n, Q_n\}$ with Q_n a projection of Y onto Z_n such that $\| Q_n \| = 1$. In this case the approximate equations reduce to

$$Mx_n - Q_n Tx_n = Q_n f, \tag{2.9}$$

while Proposition 2.5 admits the following generalization (cf. [42]).

Proposition 2.6. *Let* X *and* Y *be Banach spaces,* $M: X \to Y$ *a continuous linear injective mapping such that* $\operatorname{dist}(y, M(X_n)) \to 0$ *for each* y *in* Y, *and* $\Gamma_a = \{X_n, V_n; M(X_n), Q_n\}$ *with* $\| Q_n \| = 1$. *Suppose that*

(1) *there exists a constant* $c > 0$ *such that* $\chi(M(Q)) \geqslant c\chi(Q)$ *for any bounded set* $Q \subset X$;

(2) $T: X \to CK(Y)$ *is bounded, u.s.c., and* $\chi(T(Q)) < c\chi(Q)$ *for each bounded set* $Q \subset X$ *with* $\chi(Q) \neq 0$.

Moreover, suppose that there exists $r > 0$ such that either one of the following conditions holds.

(i) *T is odd on $X \backslash B(0, r)$.*

(ii) *Mappings K, K_n, and M_n satisfy conditions* (C1) *and* (C2) *of Proposition* 1.3 *and $(Mx - u, v) \geqslant 0$ or $(Mx - u, v) \leqslant 0$ for $u \in T(x)$, $v \in K(x)$ and $\| x \| \geqslant r$.*

Then, if $M - T$ satisfies condition $(+)$, Eq. (2.6) *is feebly approximation-solvable for each f in Y.*

Proof. In view of Proposition 1.6, it is sufficient to show that $M - T$: $X \to CK(Y)$ is A-proper w.r.t. Γ_a. Let $\{x_{n_j} \mid x_{n_j} \in X_{n_j}\}$ be a bounded sequence such that

$$g_{n_j} \equiv M(x_{n_j}) - Q_{n_j}(u_{n_j}) - Q_{n_j}(g) \to 0 \qquad \text{as} \qquad i \to \infty$$

for some g in Y and $u_{n_j} \in T(x_{n_j})$. By the construction of Γ_a and condition (1), $\chi(\{Q_{n_j}(u_{n_j})\}) \leqslant \chi(\{u_{n_j}\})$ (see [43]) and $c\chi(\{x_{n_j}\}) \leqslant \chi(\{Mx_{n_j}\}) \leqslant \chi(\{g_{n_j}\}) + \chi(\{Q_{n_j}(u_{n_j})\}) + \chi(\{Q_{n_j}(g)\}) = \chi(\{Q_{n_j}(u_{n_j})\}) \leqslant \chi(T(\{x_{n_j}\})) < c\chi(\{x_{n_j}\})$, a contradiction, unless $\{x_{n_j}\}$ is relatively compact. Here we used the fact that $Q_{n_j}(g) \to g$, which easily follows from $\| Q_n \| = 1$ and $\operatorname{dist}(y, M(X_n)) \to 0$ as $n \to \infty$ for each $y \in Y$. Consequently, for some subsequence $\{x_{n_{j(k)}}\}$ we have $x_{n_{j(k)}} \to x_0$ in X. By the upper semicontinuity of T and the continuity of M, it follows that $g \in M(x_0) - T(x_0)$. Q.E.D.

Note added in proof. Some results from [23] cited in this paper have already appeared in P. S. Milojević, A generalization of Leray-Schauder theorem and surjectivity results for multivalued A-proper and pseudo-A-proper mappings, *Nonlin. Anal., Theory, Methods and Appl.*, (1) 3 (1971), 263–276.

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